COMPUTER-AIDED DESIGN OF RECURSIVE DIGITAL NOTCH-FILTER WITH ONE COEFFICIENT

Tran Truc Viet

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

COMPUTER-AIDED DESIGN OF RECURSIVE
DIGITAL NOTCH-FILTER WITH ONE COEFFICIENT

by

Tran Truc Viet

June 1975

Thesis Advisor: S. R. Parker

Approved for public release; distribution unlimited.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Computer-Aided Design of Recursive Digital Notch-Filter with One Coefficient		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis; June 1975
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Tran Truc Viet		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE June 1975
		13. NUMBER OF PAGES 125
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Naval Postgraduate School Monterey, California 93940		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Digital Notch-Filters
State Space Equations
Round-Off Error
Optimum Problem
CCD

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A procedure for the design of a recursive digital notch-filter with one coefficient is proposed. This type of filter can be implemented readily with CCD (charge-coupled device). The effect of coefficient on notch-gain, notch-width, and passband-ripple is examined. The computer implementation is tested for second, third, and sixth order notch-filters and the optimum coefficient is found based upon limited computer wordlength.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
(Page 1) S/N 0102-014-6601 |

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Computer-Aided Design of Recursive
Digital Notch-Filter with One Coefficient

by

Tran Truc Viet

B.S., Naval Postgraduate School, 1975

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
June 1975

ABSTRACT

A procedure for the design of a recursive digital notch-filter with one coefficient is proposed. This type of filter can be implemented readily with CCD (charge-coupled device). The effect of coefficient on notch-gain, notch-width, and passband-ripple is examined. The computer implementation is tested for second, third, and sixth order notch-filters and the optimum coefficient is found based upon limited computer wordlength.

TABLE OF CONTENTS

ACKNOWLEDGEMENT

The author wishes to express his appreciation to his thesis advisor, Dr. Sydney R. Parker, for his excellent guidance. And many thanks to LCDR Lennart Souchon, German Navy, for many helpful discussions during this work.

I. INTRODUCTION

Digital filters are finding wide applications in many diverse areas, such as speech processing, picture processing, sonar and radar system, digital control system, etc. A very useful type of such filters is the notch-filter which attenuates highly a particular frequency component in the input signal while leaving nearby frequencies relatively unattenuated.

A linear digital filter can be realized physically in terms of the three basic operations of addition, multiplication and unit delay. The advantages gained by realizing a digital filter rather than its continuous counterpart lie principally in its stability, accuracy, reliability and flexibility [Refs. 7 and 20]. Furthermore, its frequency characteristics can be varied without altering the hardware simply by changing the stored set of filter coefficients. The disadvantages of digital approach lie in two extra design considerations. Firstly, the choice of wordlength of the computer and analog-to-digital converter. These problems do not arise in the corresponding continuous design. A design filter operates on an input-sampled signal by means of a computational algorithm. It can be simulated on a general purpose computer or can be constructed with special purpose digital hardware, such as CCD (charge-coupled device) implementation; this part

is beyond the the scope of research. Due to computer limited wordlength, the round-off error in computer program is produced. This can cause the filter characteristics to deviate considerably from the ideal.

This research attempts to present a survey and discussion of the effect of the finite wordlength on the accuracy of digital notch-filter. The design procedure of digital notch-filter was to propose rational functions of a discrete variable which, when transformed into the frequency domain, have desired filtering characteristics. The design was proceeded directly in terms of the variable, z, associated with discrete time intervals rather than through the intermediate variables associated with continuous time.

The digital notch-filter with one coefficient was investigated and tested by implementing on computer IBM 360/70 installed at the Naval Postgraduate School.

II. THE CHARACTERISTIC TRANSFER FUNCTION OF DIGITAL NOTCH-FILTER

A. TRANSFER FUNCTION

As the first step in the design of the digital notch-filter, one often uses the bilinear transformation to approximate the analog filters in discrete time. This transformation process is perhaps the most utilized approach relating the continuous and discrete domains. The approach in this paper differs by starting directly in the discrete domain without the bilinear transformation.

The approach is to uniformly sample the signal u(t) every T seconds, and use the resulting sequence as the input to a recursive digital filter governed by the general difference equation

$$v(K) = b_0\, u(K) + b_1\, u(K-1) + \ldots + b_m\, u(K-m)$$

$$- a_1\, v(K-1) - a_2\, v(K-2) - \ldots - a_m\, v(K-m) \qquad (2\text{-}1)$$

where u(k) and v(k) denote the values of the filter's input and output signals, respectively, at the K-th instant.

For minimum noise generating, cost, and simplicity, one can reduce equation (2-1) to the minimum number of coefficients such that (2-1) is still in the recursive form.

$$v(K) = u(K) + b_m\, u(K-m) - a_m\, v(K-m) \qquad (2\text{-}2)$$

The above simple recursive equation of the recursive digital filter is derived from (2-1) by letting

$$b_0 = 1$$

$$b_1 = b_2 = \ldots = b_{m-1} = 0$$

and $$a_1 = a_2 = \ldots = a_{m-1} = 0$$

Then its corresponding transfer function has the form

$$H(z) = \frac{1 + b_m z^{-m}}{1 + a_m z^{-m}} \tag{2-3}$$

For a notch-filter one wishes the filter characterized by (2-3) to effectively transmit all sinusoidal inputs with the exception of the ω_0 radiant/sec. sinusoid which it should completely reject. Unfortunately, it is impossible to implement an ideal notch-filter

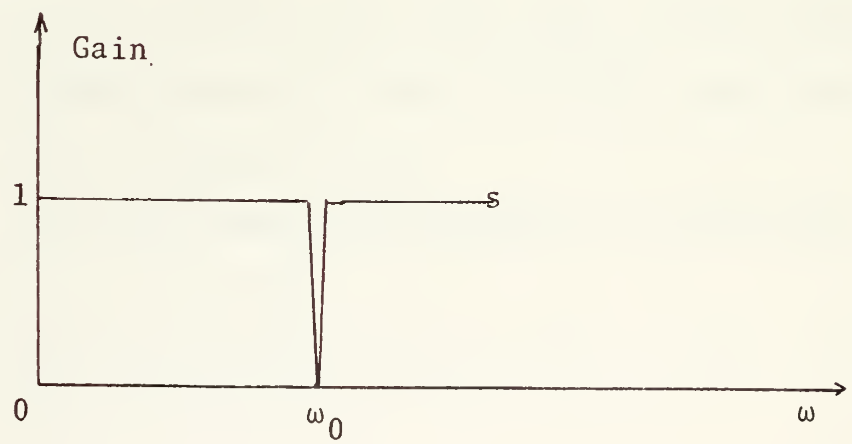


Figure (2-1) Gain frequency behavior of an ideal notch filter.

B. GAIN FACTOR OF A DIGITAL FILTER

The poles and zeros of the transfer function (2-3) are computed as follows

$$z^m + b_m = 0$$

hence $\quad z = \sqrt[m]{-b_m} = \sqrt[m]{b_m}\; e^{j(2n+1)\frac{\pi}{m}}$

and $\quad z^m + a_m = 0$

$$n=0,1,2,\ldots$$

$$z = \sqrt[m]{-a_m} = \sqrt[m]{a_m}\; e^{j(2n+1)\frac{\pi}{m}}$$

The zeros and poles are respectively given by

$$z_i = \sqrt[m]{b_m}\; e^{j(2n+1)\frac{\pi}{m}} \tag{2-4}$$

and

$$p_i = \sqrt[m]{a_m}\; e^{j(2n+1)\frac{\pi}{m}} \tag{2-5}$$

$$i=1,\ldots,m.$$

From (2-4) and (2-5) the zeros and poles are all complex conjugate for m even and for m odd, one of the zeros and one of the poles will be negative real.

Let m=3, the zero's and pole's pattern is plotted in Figure (2-2).

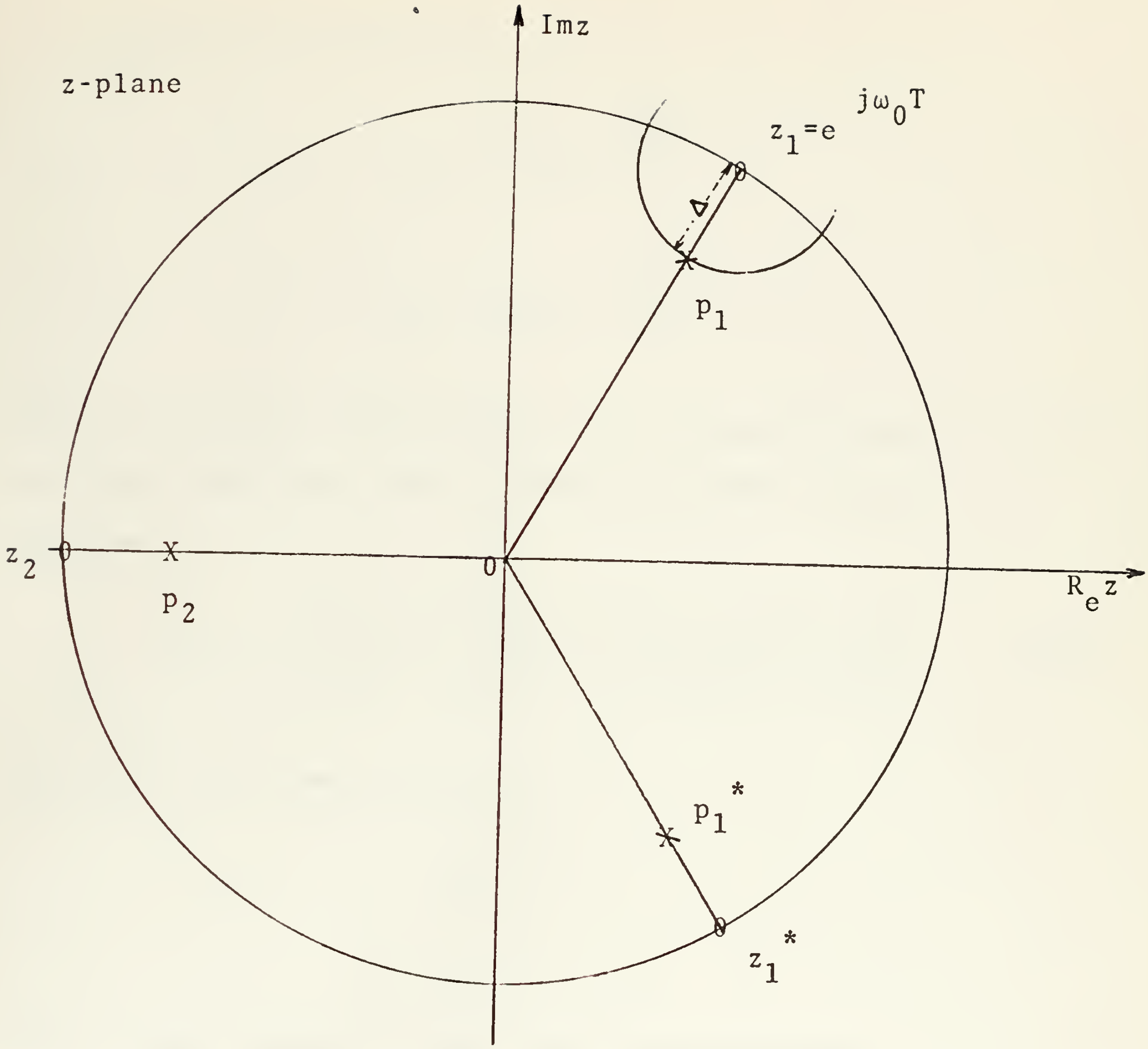


Figure (2-2) Pole-zero pattern.

where

z_1^* is complex conjugate of z_1,

T is sampling frequency,

ω_0 is a notch-frequency which the one in which the first zero and pole occur.

For m=3 it is given by

$$z_1 = \sqrt[3]{b_3} \quad e^{j\frac{\pi}{3}}$$

$$p_1 = \sqrt[3]{a_3} \quad e^{j\frac{\pi}{3}}$$

After multiplying the numerator and the denominator of (2-3) by z^m and factoring, the following zero-pole format is obtained

$$H(z) = \frac{(z-z_1)\ (z-z_1^*)\ \ldots\ (z-z_{2\ell})\ (z-z_{2\ell}^*)}{(z-p_1)\ (z-p_1^*)\ \ldots\ (z-p_{2\ell})\ (z-p_{2\ell}^*)} \tag{2-6a}$$

where m is even and given by

$$m = 2\ell$$

$$\ell = 1,\ 2,\ \ldots,\ N.$$

If m is odd, called $m = 2\ell+1$, the format becomes

$$H(z) = \frac{(z-z_1)\ (z-z_1^*)\ \ldots\ (z-z_{2\ell})\ (z-z_{2\ell}^*)\ (z-z_{2\ell+1})}{(z-p_1)\ (z-p_1^*)\ \ldots\ (z-p_{2\ell})\ (z-p_{2\ell}^*)\ (z-p_{2\ell+1})} \tag{2-6b}$$

where $z_{2\ell+1}$ and $p_{2\ell+1}$ are negative real.

The condition for the system being stable is:

$$\sum_{\omega=0}^{\infty} |H\ (e^{j\omega\ T})| < \infty \tag{2-7}$$

and the steady-state response of any stable system with transfer function (2-6) to the sinusoidal input, $u(K) = \text{Sin } K\omega T$, is given by

$$y(K) = |H\ (e^{jK\omega T})|\ \text{Sin}\ (K\omega T + \theta)$$

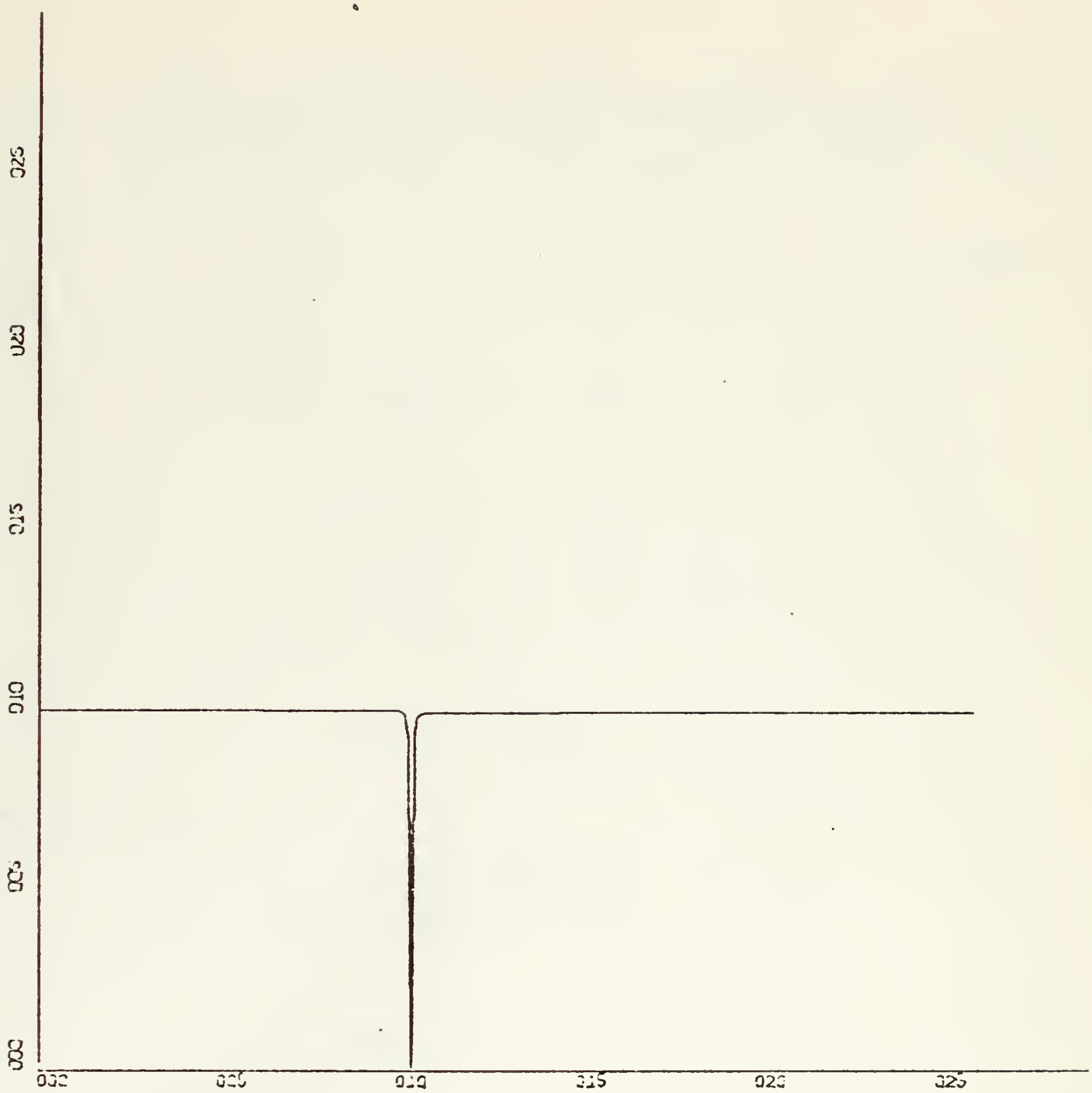
where

$$H\ (e^{j\omega T}) = H\ (z)|_{\ z\ =\ e^{j\omega T}}$$

and

$$\theta = \text{angle}\ \{H\ (e^{j\omega T})\}\ .$$

The term $|H\ (e^{j\omega T})|$ is called the system gain factor since it characterizes the sinusoidal gain response property of system (2-6). Hence, the steady state sinusoidal response is readily obtained by evaluating the system transfer function at $z=e^{j\omega T}$, where ω is the radian frequency of the input sinusoid and T is the sampling period. One can determine the system's gain and phase angle factor in a straightforward manner by using a geometrical approach. This approach gives the advantage of relationship between the locations of zeros and poles of the system transfer function and its gain-factor behavior. In the design of digital filter, it will enable us to design a system whose gain-factor approximates any desired sinusoidal discrimination behavior. Substituting $z=e^{j\omega T}$ into (2-6b),



X-SCALE=5.00E-01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

Figure (2-5) Frequency magnitude plot for m=3.

$$H(e^{j\omega T}) = \frac{(e^{j\omega T}-z_1)(e^{j\omega T}-z_1^*)\ldots(e^{j\omega T}-z_{2\ell})(e^{j\omega T}-z_{2\ell}^*)(e^{j\omega T}-z_{2\ell+1})}{(e^{j\omega T}-p_1)(e^{j\omega T}-p_1^*)\ldots(e^{j\omega T}-p_{2\ell})(e^{j\omega T}-p_{2\ell}^*)(e^{j\omega T}-p_{2\ell+1})} \tag{2-8}$$

where

$$z_i = \sqrt[m]{b_m}\, e^{j(2n+1)\frac{\pi}{m}}$$

$$i = 1, 2, \ldots, 2\ell$$

$$n = 0, 1, \ldots, 2\ell$$

$$m = 2\ell + 1$$

$$z_{2\ell+1} = -\sqrt[m]{b_m}\,.$$

Similarly

$$p_i = \sqrt[m]{a_m}\, e^{j(2n+1)\frac{\pi}{m}}$$

$$p_{2\ell+1} = -\sqrt[m]{a_m}$$

let

$$\alpha_i = e^{j\omega T} - \sqrt[m]{b_m}\, e^{j(2n+1)\frac{\pi}{m}}$$

$$\beta_i = e^{j\omega T} - \sqrt[m]{a_m}\, e^{j(2n+1)\frac{\pi}{m}}$$

$$\alpha_{2\ell+1} = e^{j\omega T} + \sqrt[m]{b_m}$$

$$\beta_{2\ell+1} = e^{j\omega T} + \sqrt[m]{a_m} \quad .$$

Hence (2-8) becomes

$$|H(e^{j\omega T})| = \frac{|\alpha_1|\ |\alpha_1^*|\ \ldots\ |\alpha_{2\ell}|\ |\alpha_{2\ell}^*|\ |\alpha_{2\ell+1}|}{|\beta_1|\ |\beta_1^*|\ \ldots\ |\beta_{2\ell}|\ |\beta_{2\ell}^*|\ |\beta_{2\ell+1}|} \qquad (2\text{-}9)$$

It is obviously the system gain factor at radian frequency ω is seen to depend on the distance by which the system transfer function's zeros and poles are located from the critical point $e^{j\omega T}$. Thus, if one wishes to completely reject the sinusoid sin $k\omega T$, it is necessary that the transfer function have the radian frequency ω increases from zero to infinitive, one sees that the zero of transfer function (2-3) will fall at the first of the zeros of $[(2n+1)\frac{\pi}{m}]$ radians for n=0 and repeats for n=1,2,... as in Fig. (2-3).

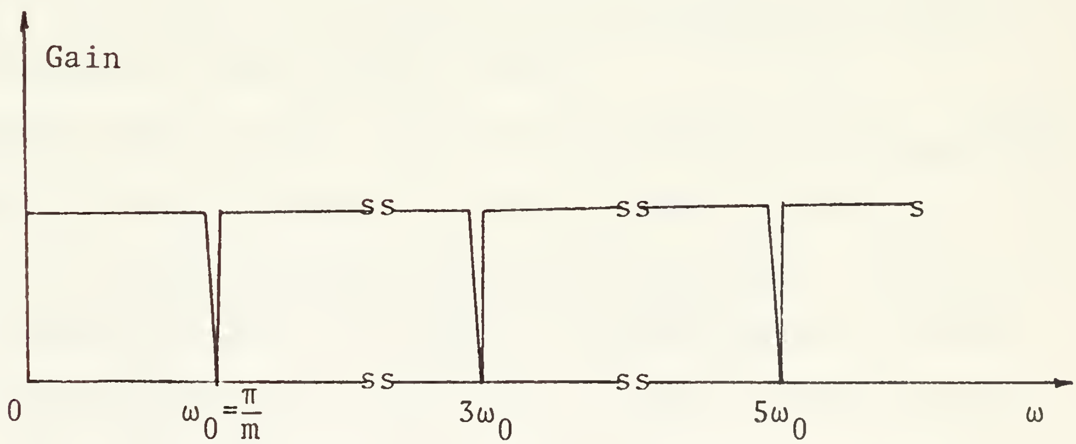


Figure (2-3) Gain frequency behavior of digital notch-filter.

By considering pole-zero pattern in Fig. (2-2) and the equation (2-9) one wishes to reject completely at $z = e^{j(2n+1)\frac{\pi}{m}}$, the distance α_i must be equal to zero.

One knows that

$$\alpha_i = z - \sqrt[m]{b_m}\, e^{j(2n+1)\frac{\pi}{m}}$$

hence

$$\alpha_i = e^{j(2n+1)\frac{\pi}{m}} - \sqrt[m]{b_m}\, e^{j(2n+1)\frac{\pi}{m}} .$$

From the above equation,

$$\alpha_i = 0 \qquad \text{implies } b_m = 1$$

and for the gain of (2-9) is constant for all frequencies except one at $z=e^{j(2n+1)\frac{\pi}{m}}$. One should choose the coefficient, a_m, as close to unity as possible; the closer a_m is to unity the closer the notch-filter approaches the ideal notch-filter as shown in Fig. (2-1).

Looking back in Fig. (2-2), the parameter $\Delta = 1-a_m$, which controls the notch-bandwidth, when a_m close to one, Δ close to zero and so does the notch-bandwidth. Ideally, it is desired to make Δ as small as possible, but in practice considerations one is not able to make Δ arbitrarily small (Δ will be discussed in section D of this chapter).

The above considerations can be made clear by analyzing the transfer function (2-3).

C. ANALYSIS OF MAGNITUDE RESPONSE

It is discussed here the magnitude response as a function of frequency and the coefficients a_m and b_m.

The equation (2-3) is rewritten as follows:

$$H(z) = \frac{z^m + b_m}{z^m + a_m} \tag{2-10}$$

Let $z=e^{j\omega T}$ and substitute into (2-10), it is given by

$$H(e^{j\omega T}) = \frac{e^{jm\omega T} + b_m}{e^{jm\omega T} + a_m} \tag{2-11}$$

Using $e^{jx} = \cos x + j\sin x$, one gets

$$H(e^{j\omega T}) = \frac{(\text{Cos } m\omega T + b_m) + j \sin m\omega T}{(\text{Cos } m\omega T + a_m) + j \sin m\omega T} \tag{2-12}$$

The squared magnitude of $H(e^{j\omega T})$ will be

$$|H(e^{j\omega T})|^2 = \frac{(\text{Cos } m\omega T + b_m)^2 + \sin^2 m\omega T}{(\text{Cos } m\omega T + a_m)^2 + \sin^2 m\omega T}$$

or

$$|H(e^{j\omega T})|^2 = \frac{1 + b_m^2 + 2b \text{ Cos } m\omega T}{1 + a_m^2 + 2a \text{ Cos } m\omega T} \tag{2-13}$$

Take derivative of (2-13) with respect to ω and let the numerator equal to zero

$$- 2mb_mT \sin m\omega T \, [1 + a_m^2 + 2a_m \cos m\omega T]$$

$$+ 2ma_mT \sin m\omega T \, [1 + b_m^2 + 2b_m \cos m\omega T] = 0$$

or

$$(-2mb_mT)(1+a_m^2) \sin m\omega T + 2ma_mT \, (1+b_m^2) \sin m\omega T = 0$$

Hence

$$\sin m\omega T \, [a_m(1 + b_m^2) - b_m(1 - a_m^2)] = 0 \qquad (2\text{-}14)$$

Suppose the term in the bracket is not equal to zero. The frequency at which the squared magnitude is maximum or minimum will satisfy the equation (2-14)

$$\sin m\omega T = 0 \qquad \text{implies} \qquad \omega T = \frac{n\pi}{m}$$

$$n=0,1,2,...$$

At n=0, one has $\omega T=0$, the magnitude has the maximum value and at n=1, $\omega T=\frac{\pi}{m}$, the magnitude has the minimum value and $\omega=\frac{\pi}{m}$ is defined as a notch-frequency; when n=2,3,.... the magnitude function just repeats as in Fig. (2-4).
If one wishes to get the constant gain except at frequency $\omega=n\frac{\pi}{mT}$, n=1,3,5,..., as in Fig. (2-3), the coefficients a_m and b_m are selected such that this result is achieved.
To do that in equation (2-14), one lets the factor in the bracket always equal zero, i.e.,

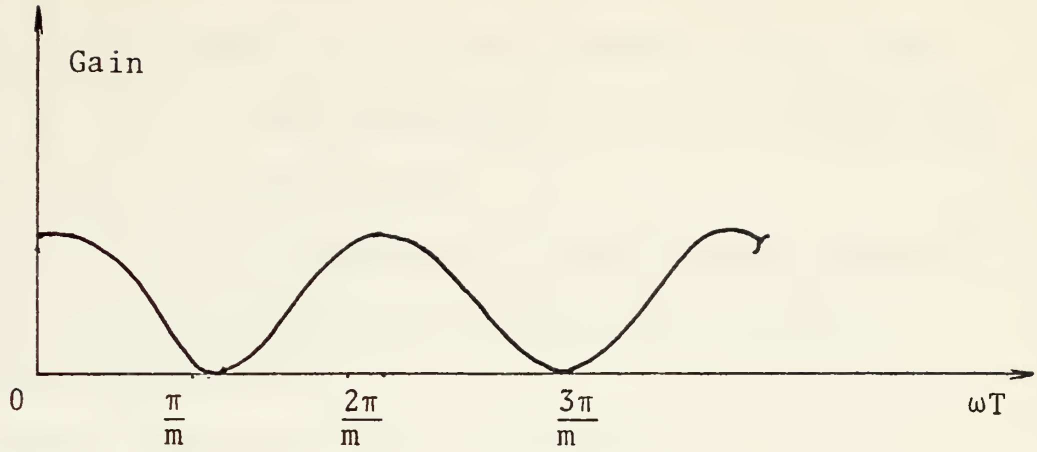


Figure (2-4) Gain frequency behavior with some values of a_m and b_m.

$$a_m\ (1+b_m^2) - b_m\ (1+a_m^2) = 0$$

or

$$a_m + a_m b_m^2 - b_m - b_m a_m^2 = 0$$

$$a_m - b_m = a_m b_m\ (a_m - b_m) \qquad (2\text{-}15)$$

Hence $$a_m b_m = 1$$

We know that the system is stable when all the roots of characteristic polynomial or transfer function of system are inside the unit circle of z-plane.

By that condition, the coefficient a_m must be

$$a_m < 1\ .$$

As stated in section B, the coefficient b_m has the value of one for completely rejecting at the notch-frequency. One

sees that with $b_m=1$ and $a_m < 1$, the equality (2-15) cannot be satisfied. But we can approximate (2-15) by letting the coefficient a_m very close to one.

One sees that both statements by analyzing the transfer function (2-3) and considering the pole-zero pattern in Fig. (2-2) are appropriate.

The transfer function (2-3) now becomes:

$$H(z) = \frac{1 + z^{-m}}{1 + az^{-m}} \tag{2-16}$$

where the subscript m is deleted and its squared magnitude is given by:

$$|H(e^{j\omega T})|^2 = \frac{2 + 2 \operatorname{Cos} m\omega T}{1 + a^2 + 2a \operatorname{Cos} m\omega T} \tag{2-16a}$$

The plot of $|H(e^{j\omega T})|$ for particular value of a vs radian frequency ω for m=3 in Fig. (2-5). The half power frequencies are that at which the gain is 3dB down or

$$|H(e^{j\omega T})| = \frac{1}{2}$$

or

$$\frac{2 + 2 \operatorname{Cos} m\omega T}{1 + a^2 + 2a \operatorname{Cos} m\omega T} = \frac{1}{2} \tag{2-17}$$

Solve (2-17) for ω

$$4 + 4 \operatorname{Cos} m\omega T = 1 + a^2 + 2a \operatorname{Cos} m\omega T$$

$$(4-2a) \operatorname{Cos} m\omega T = a^2 - 3$$

Hence

$$\cos m\omega T = \frac{a^2 - 3}{4 - 2a} \qquad (2\text{-}18)$$

There are two solutions $\omega_1 T$ and $\omega_2 T$ which are symmetric to $\omega T = \frac{\pi}{m}$, the notch-frequency of filter, as illustrated in Fig. (2-5)

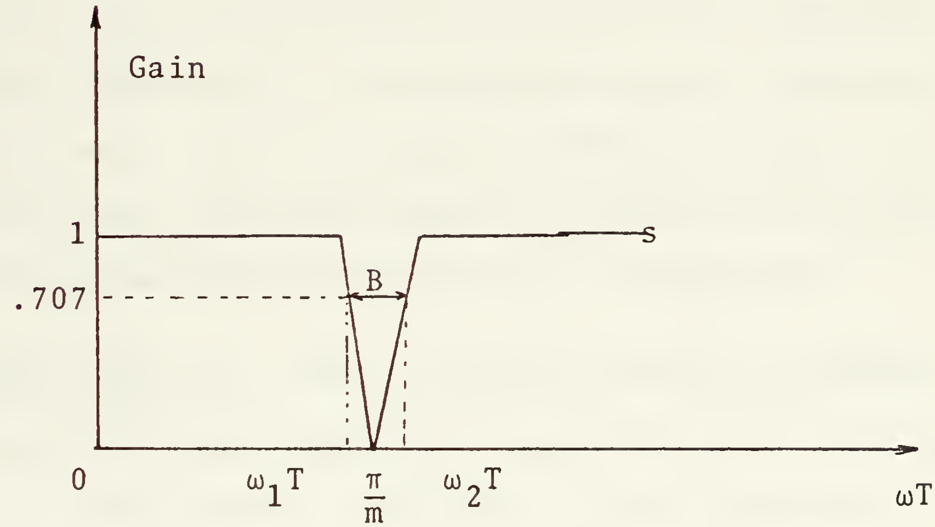


Figure (2-6)

where B is a notch-bandwidth.

The notch-bandwidth B will be

$$\frac{B}{2\pi} = T(\omega_2 - \omega_1) = 2\left[\frac{\pi}{m} - \frac{1}{m}\cos^{-1}\left(\frac{a^2 - 3}{4 - 2a}\right)\right]$$

or

$$B = \frac{4\pi}{m}\left[\pi - \cos^{-1}\left(\frac{a^2 - 3}{4 - 2a}\right)\right] \qquad (2\text{-}19)$$

If one inserts the parameter $\Delta=1-a$ into equation (2-19), the bandwidth will be a function of Δ

$$B = \frac{4\pi}{m} [\pi - \mathrm{Cos}^{-1} \frac{(1 - \Delta)^2 - 3}{4 - 2(1 - \Delta)}]$$

or

$$B = \frac{4\pi}{m} [\pi - \mathrm{Cos}^{-1} \frac{\Delta^2 - 2\Delta - 2}{2\Delta + 2}] \qquad (2\text{-}20)$$

From (2-19) and (2-20), the limit of the bandwidth B when the coefficient, a, approaches one (or Δ approaches zero) is zero. This is valid in theory; in fact, there is some limit for the coefficient, a, (also for Δ) in practical view (this phenomenon will be seen in Chapter V).

D. LOWER BOUND OF Δ DUE TO FINITE PRECISION ARITHMETIC

Because of the limitations caused by finite wordlength in all current digital computers, a real number must be represented by using a finite number of bits in either the fixed-point or floating-point form. The error introduced by such a representation is discussed in this section. Only binary arithmetic will be discussed here.

In a fixed-point representation, assume a number x has been normalized such that $|x| \leq 1$ can be represented in twos complement form as

$$x = b_0 + \sum_{i=1}^{\infty} b_i 2^{-i}$$

where

$b_i=1$ or 0 and b_0 is the sign bit.

To approximate x by a t-bit register length, one of two processes is generally utilized: rounding or chopping.

In rounding, a 1 or a 0 is first added to the t-th bit, $(x)_{t-1}$, according to whether the (t+1)-th bit, $(x)_t$, is 1 or 0. Then only the first t bits of the result are kept. In chopping, those bits beyond the most significant t bits are simply dropped. Since the error introduced by chopping is larger than that introduced by rounding, chopping arithmetic is not commonly used. Therefore, the discussion below is limited to the rounding arithmetic process.

Let $(x)_t$ be the t-bit representation of the number x. The error which will be introduced is defined by

$$\varepsilon = (x)_t - x$$

where
$$|\varepsilon| \leq 2^{-t} \tag{2-21a}$$

It is known that if two t-bit fixed-point numbers are added, their sum still contains t bits, provided that there is not overflow. Therefore, under this assumption of no overflow, fixed-point addition causes no error. On the other hand, the product of two t-bit numbers may have more than t-bits. Thus, rounding is necessary if only t bits are to be kept for future processing.

In the case of a floating-point number, x is represented as

$$x = f2^e$$

where f, a fraction between $\frac{1}{2}$ and 1, is called the mantissa and e is a binary integer called the exponent. Let x denote the t-bit mantissa floating-point approximation of the number x. The difference between x and $fl(x)=(x)_t$ is called the round-off error and depends on the size of x. It is therefore the best measure relative to x. Then it is clear that

$$(x)_t = x(1+\varepsilon) \tag{2-21b}$$

where the relative error ε is bounded by $-2^{-t} \leq \varepsilon \leq 2^{-t}$ Unlike the fixed-point case, both addition and multiplication in floating-point can introduce round-off error.

From the preceding section, it is apparent that a digital notch filter with notch at $\omega_0=\frac{\pi}{mT}$ can be implemented by means of a filter whose transfer function has a zero located at $e^{j\omega_0 T}$. In order that the gain of such a filter in the frequency domain is near one for frequencies sufficiently close to ω_0, it is necessary that there exist properly located poles which tend to cancel the effects of these zeros. This can be accomplished by selecting an appropriate Δ; from equation (2-20) it is obvious that the notch of the filter is directly dependent on the parameter Δ. Ideally, it is desirable that Δ be made as small as possible in order to make the notch width small.

There are practical considerations, however, which when taken into account, will not enable the filter designer to make Δ arbitrarily small (see Chapter V). In the design

of a notch-filter of (2-16), one wishes the gain factor at $\omega = \frac{\pi}{mT}$ to be zero. For the critical point $e^{j\omega T}$ sufficiently outside the circle of center $z_1 = e^{j\frac{\pi}{m}}$ and of radius Δ, Fig. (2-6), the gain factor for all practical purposes is one.

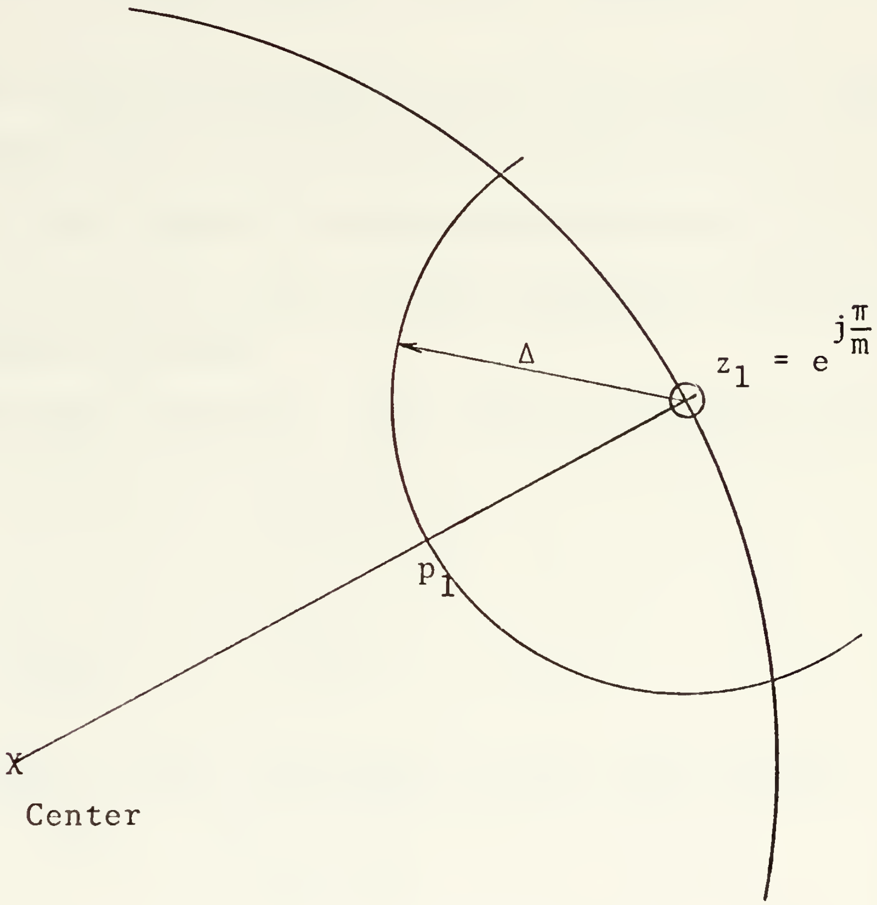


Figure (2-6) A pair of first pole and zero.

This follows since, for $z=e^{j\omega T}$ outside this circle, by selecting Δ small enough, the ratio of the product of the distances from zeros and poles to the critical point $e^{j\omega T}$ is effectively a constant. Since all digital computers

are limited by finite wordlengths, the desired coefficients will be truncated and the coefficient inaccuracies will result as stated at the beginning of the D of Chapter II.

The characteristic difference equation of a digital filter (2-16) is given by

$$v(K) = u(K) + u(K-m) - av(K-m) \tag{2-22}$$

When one wishes to implement (2-22) on a digital computer, the coefficient, a, will be stored with error due to truncation as $(a+\varepsilon)$, where ε is the round-off error in (2-21a,b). This will rise to a shift in the actual pole locations of the transfer function.

For system stability, the condition placed on $(a+\varepsilon)$ is:

$$a + \varepsilon < 1$$

or $$1-a > \varepsilon$$

hence $$\Delta > \varepsilon \tag{2-23a}$$

If one uses a t-bit wordlength, (2-23) will become:

$$\Delta > 2^{-t} \tag{2-23b}$$

In the implementation of digital notch-filter (2-22) with t-bit wordlength, one must select the parameter $\Delta > 2^{-t}$ to guarantee that the notch-width is sufficiently narrow and to avoid the possibility of unbounded system growth. It is noted that if one counts on guard bit, the wordlength will be (t+1) bits. Given notch-width B, one can calculate Δ from (2-20) and therefore t.

III. STATE SPACE REPRESENTATIONS AND REALIZATIONS

The transfer function (2-16) of an m-th order digital filter is the relationship between one input and one output. Only state and output equations with single input and single output will be considered. In the time invariant discrete system, the transfer function (2-16) can be described by a state variable representation of the form

$$x(K+1) = A\ x(K) + B\ u(K) \qquad (3\text{-}1a)$$

$$v(K) = C\ x(K) + \delta\ u(K) \qquad (3\text{-}1b)$$

where

A is an m x m matrix,

B is an m x 1 matrix

C is a 1 x m matrix

δ is a 1 x 1 matrix or scalar.

Taking the z-transform of (3-1a) and solving for X(z) yields

$$X(z) = [I - Az^{-1}]^{-1}\ B\ U(z) \qquad (3\text{-}2)$$

where I is an m-dimension identity matrix. Then, taking the z-transform of equation (3-1b) and substituting (3-2) for X(z) yields

$$V(z) = \{C\ [I - Az^{-1}]^{-1}\ B + \delta\}\ U(z)$$

$$= H(z)\ U(z) \qquad (3\text{-}3)$$

where $$H(z) = C\,[I - Az^{-1}]^{-1}\,B + \delta \qquad (3\text{-}4)$$

H(z) is the transfer function of system (3-1). The discrete system is considered, as stated above, for only one input and one output. Hence, the matrix parameters A, B, C and δ have the forms:

$$\underset{m \times m}{A} = \begin{bmatrix} a_{11} & \cdots\cdots & a_{1m} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{m1} & \cdots\cdots & a_{m \times m} \end{bmatrix}$$

$$\underset{m \times 1}{B} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad (3\text{-}5)$$

$$\underset{1 \times m}{C} = \begin{bmatrix} [C_1 & \cdots\cdots & C_m] \end{bmatrix}$$

$$\delta = d$$

There are many ways to realize the digital filter (3-1). Some of them are the canonic realizations. The term "canonic" is used in the sense of minimizing the number of operations required. This minimization implies a minimum number of coefficients of the difference equations, which is the source of noise.

If one wants to describe the system (3-1) by a matrix form as:

$$S = \left(\begin{array}{c|c} A & B \\ \hline C & \delta \end{array}\right) \qquad (3\text{-}7)$$

where S is called characteristic matrix system.

The matrix parameters A, B, C and δ are respectively given as follows:

$$A = \left[\begin{array}{ccc|c} 0 & \cdots & \cdots & -a \\ \hline & & & 0 \\ & & & \cdot \\ & (m-1)\times(m-1) & & \cdot \\ & I & & \cdot \\ & & & \cdot \\ & & & 0 \end{array}\right]$$

$$B = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad (3\text{-}6)$$

$$C = [\tfrac{a-1}{a} \quad 0 \; . \; . \; . \; 0]$$

$$\delta = \frac{1}{a}$$

To describe a system in the matrix form, its state equations will have the form:

$$\begin{aligned}
x_1(K+1) &= -a\, x_m(K) + u(K) \\
x_2(K+1) &= x_1(K) \\
&\;\;\vdots \\
x_m(K+1) &= x_{m-1}(K) \\
v(K) &= \left(\frac{a-1}{a}\right) x_1(K) + \frac{1}{a}\, u(K)
\end{aligned} \qquad (3\text{-}7)$$

The corresponding realization of (3-7) has the form

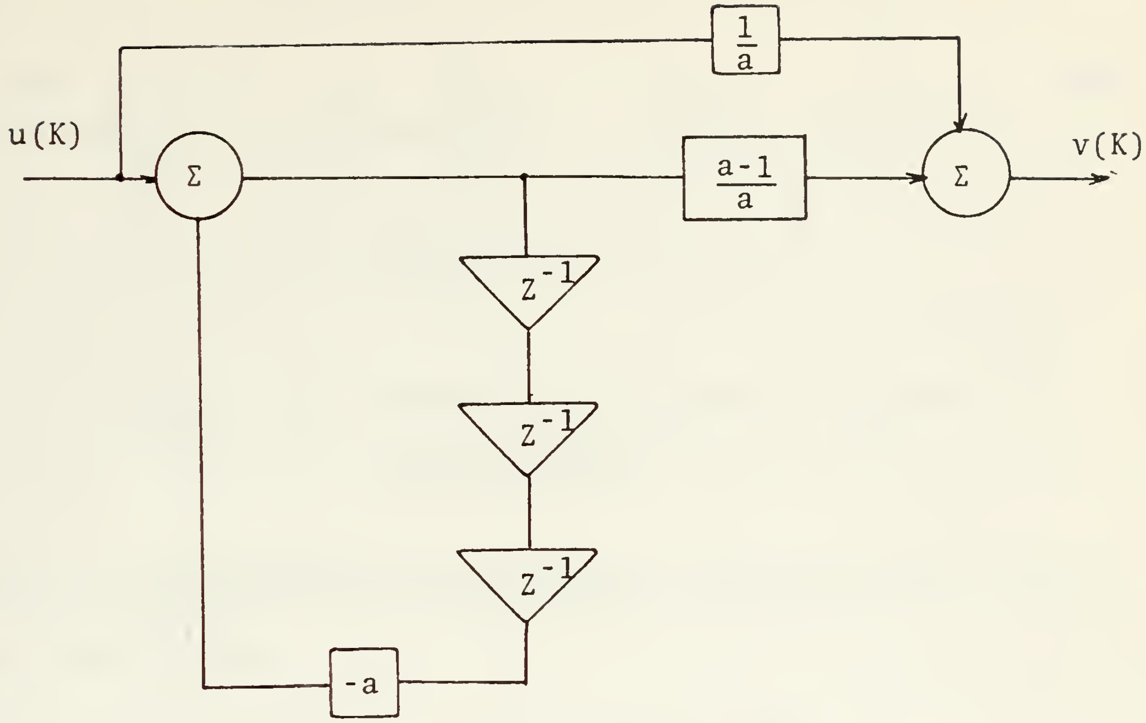


Figure (3-1) Non-canonic realization of m-th order digital notch-filter.

The realization in Fig. (3-1) is not canonic; the purpose of this paper is to use only one coefficient. There are many disadvantages in the above realization. Among these are the production of more noise, higher cost and greater network complexity. Hence, the matrix form (3-7) is a poor way to describe the system. This leads to the consideration of a search technique for the canonic realization as stated in section A. From this concept, one can realize the digital notch-filter (2-6) as follows:

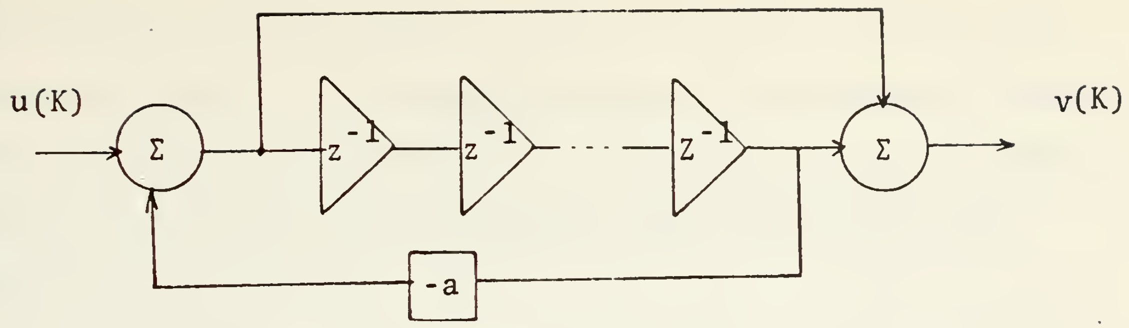


Figure (3-2) Canonic realization of digital notch-filter.

and the corresponding state equations of realization (3-1) are listed below

$$
\begin{aligned}
x_1(K) &= -a\, x_m(K+1) + u(K) \\
x_1(K+1) &= x_1(K) \\
x_2(K+1) &= x_1(K+1) \\
&\ \ \vdots \\
x_m(K+1) &= x_{m-1}(K+1) \\
v(K) &= x_m(K+1) + x_1(K)
\end{aligned}
\qquad (3\text{-}8)
$$

One sees the realization in Fig. (3-2); with this technique there is only one coefficient and it provides the minimum number of the coefficients of the m-th order recursive digital filter one can achieve. Therefore, in noise analysis, if one consider only the source of noise produced by the multiplications of the coefficients, it is not necessary to assume the sources of noise are

uncorrelated, since there is only one source of noise in the realization. By this advantage, in the design of high order of digital notch-filter it is not necessary to implement by cascading first or second order sections as is often done at present.

The "dual" canonic realization of Fig. (3-2) can be derived. The so-called "transposed realization" has been shown by Jackson [Ref. 4] to be given in the form

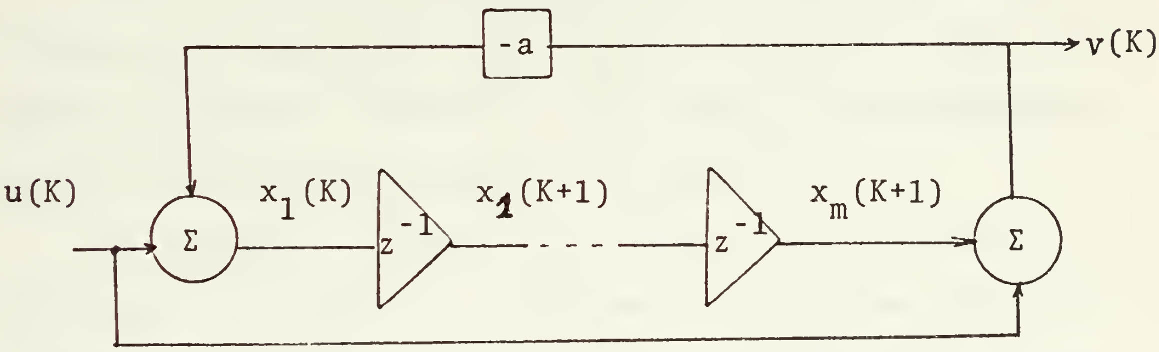


Figure (3-3) Transposed canonic realization of digital notch-filter.

The corresponding equations are given by:

$$
\begin{aligned}
x_1(K) &= -a\,u(K) + u(K) \\
x_1(K+1) &= x_1(K) \\
x_2(K+1) &= x_1(K+1) \\
&\;\;\vdots \\
x_m(K+1) &= x_{m-1}(K+1) \\
v(K) &= x_m(K+1) + u(K)
\end{aligned}
\qquad (3\text{-}9)
$$

In the state equations (3-8) and (3-9), the value of the coefficient, a, is taken between 0 and 1. It is known that the notch-gain and notch-width of a digital notch-filter are functions of the coefficient a. The question which arises here is: what is the optimum value of the coefficient a which satisfies the desired characteristics of the notch-filters and how large is the ripple in their passband?

These questions are investigated in the following chapters by simulating the digital notch-filter with the digital computer IBM 360/70. One can see the difference between theory and practice in Chapter III.

For example, the gain in equation (2-9) at notch frequency, in theory, is a decreasing function when the coefficient a is increasing. However, in practice, some values of coefficient a make the notch-gain change non-monotonically because of round-off noise in iteration computations in the program.

IV. IMPLEMENTATION

A. COMPUTER IMPLEMENTATION

In this chapter, a digital notch-filter which is characterized by the state equations (3-8) is implemented on the IBM 360/70. Plots of the gain-frequency response and the time response are provided.

The input signal is a sine-wave, i.e., a signal of single frequency, which sampled every T seconds. The corresponding gain for each input sine-wave is computed when the steady state is reached, by the ratio of the value of the peak input sine-wave divided by the value of peak output response. Then the input sine-wave is scanned over the frequency range of the notch-filter and the gain-frequency response is plotted. It is noted that the notch of the filter falls at the first zero of filter; that is, at $z_1 = e^{j\frac{\pi}{m}}$. If one desires the notch-filter at ω_0 radians, the sampling time T is computed as follows

$$e^{j\omega_0 T} = e^{j\frac{\pi}{m}}$$

or

$$\omega_0 T = \frac{\pi}{m}$$

Hence

$$T = \frac{\pi}{m\omega_0}$$

or $$T = \frac{1}{2mf_0} \qquad (4\text{-}1)$$

where $f_0 = \frac{\omega_0}{2\pi}$ is the notch-frequency in hertz.

The following sections contain implementations of second, third, and sixth order digital notch-filters on the IBM 360 computer using single precision arithmetic.

1. Second Order Digital Notch-Filter

The results of m-th order digital notch-filter in Chapters II and III are applied. The second order digital notch-filter is characterized by the transfer function

$$H(z) = \frac{1 + z^{-2}}{1 + az^{-2}} \qquad (4\text{-}2)$$

and it is realized as follows:

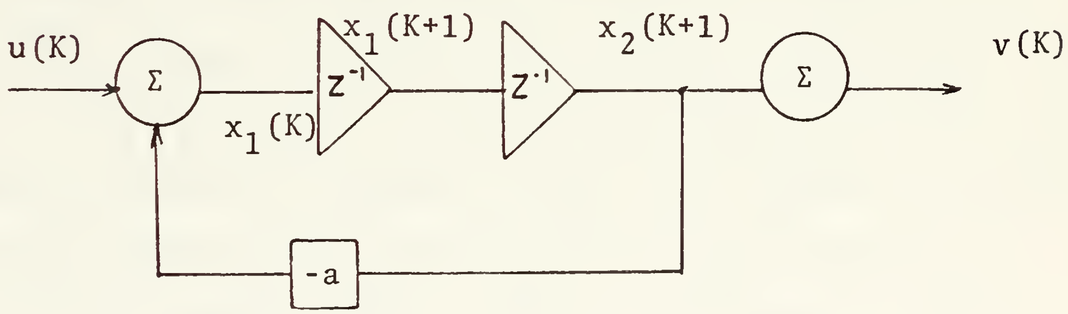


Figure (4-1) Canonic realization of second order digital notch-filter.

The corresponding state equations are given by:

$$
\begin{aligned}
x_1\ (K) &= -a\ x_2\ (K+1) + u\ (K) \\
x_1\ (K+1) &= x_1\ (K) \\
x_2\ (K+1) &= x_1\ (K+1) \\
v\ (K) &= x_2\ (K+1) + x_1\ (K)
\end{aligned}
\qquad (4\text{-}3)
$$

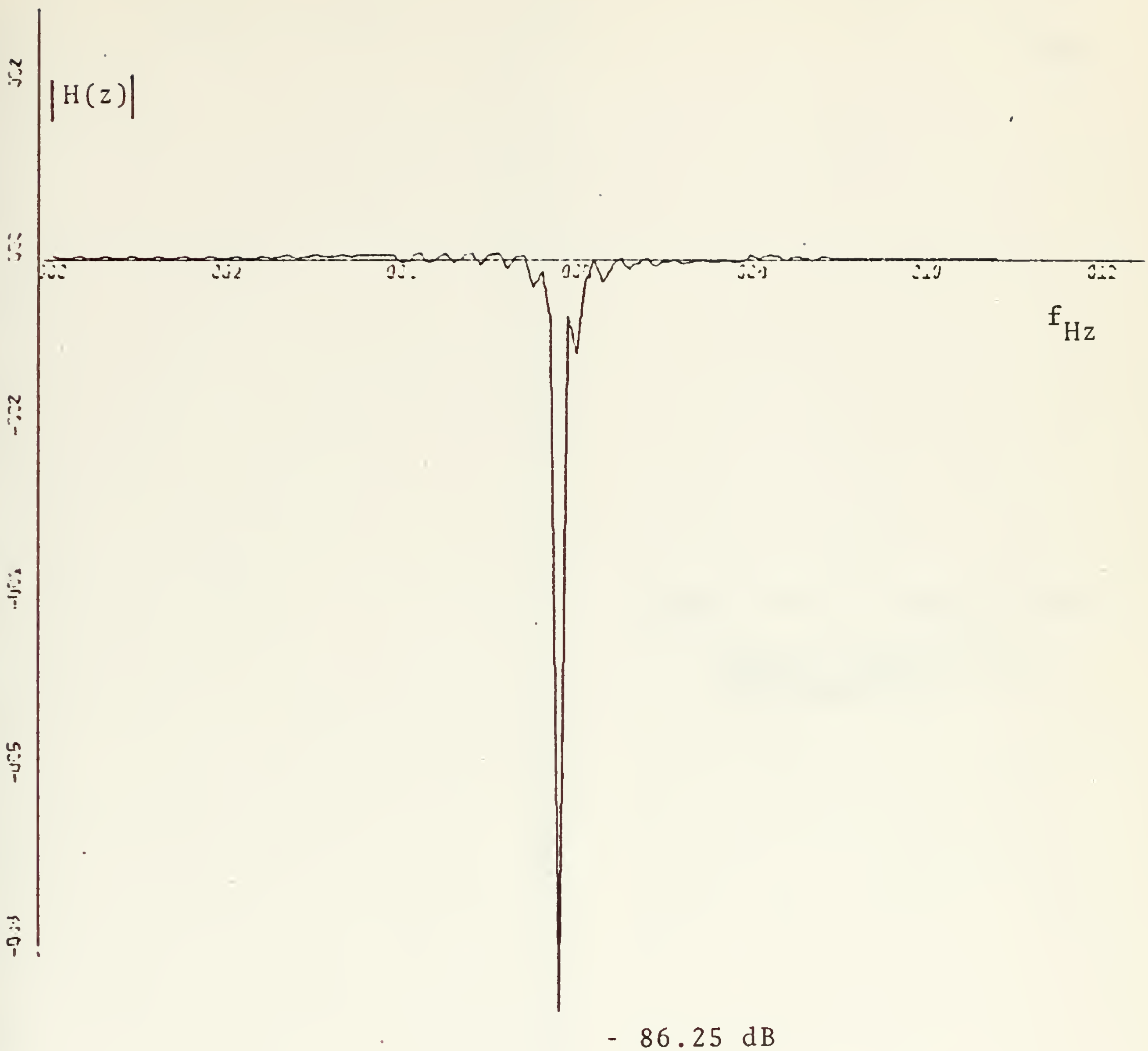
If one wishes a notch frequency at $f_0 = 60$ hz, the equation (4-1) gives the sampling frequency T as follows

$$
T = \frac{1}{2mf_0} = \frac{1}{2 \times 2 \times 60} = 4.166666667 \times 10^{-3}
$$

The figures from (4-2) to (4-10) illustrate the gain frequency responses in dB with different coefficients of one thousand iterations. It is noted that at one thousand iterations the steady state of frequency response is not yet reached. These plots merely depict the behavior of the gain-frequency response with changing coefficients.

The result of the simulating program gives the steady state of frequency response at about six thousand iterations. They are shown in Figs. (4-2) to (4-17) corresponding to some typical coefficients. And the notch-gains, notch-widths, and maximum passband-ripple in dB of the steady state gain-frequency response are tabulated in Table (4-1). Fig. (4-11) shows the plot of notch-width vs. coefficient in theory by evaluating the equation (2-19).

The time response of a second order digital notch-filter when the input is a sinusoidal signal of frequency 60 Hz, i.e., at the notch frequency given below. In theory, the notch-filter with 60 Hz notch will immediately reject the above input signal, but in practice the time response sinusoidal signal will be damped in some finite time, small or large according to the coefficient and order of notch-filters one utilizes. Figures (4-18) to (4-21) show the time response of second order 60 Hz notch-filters with 60 Hz sinusoidal input signal applied.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+01 UNITS INCH.

Figure (4-2) Frequency response of second order digital notch-filter. a=0.90714359

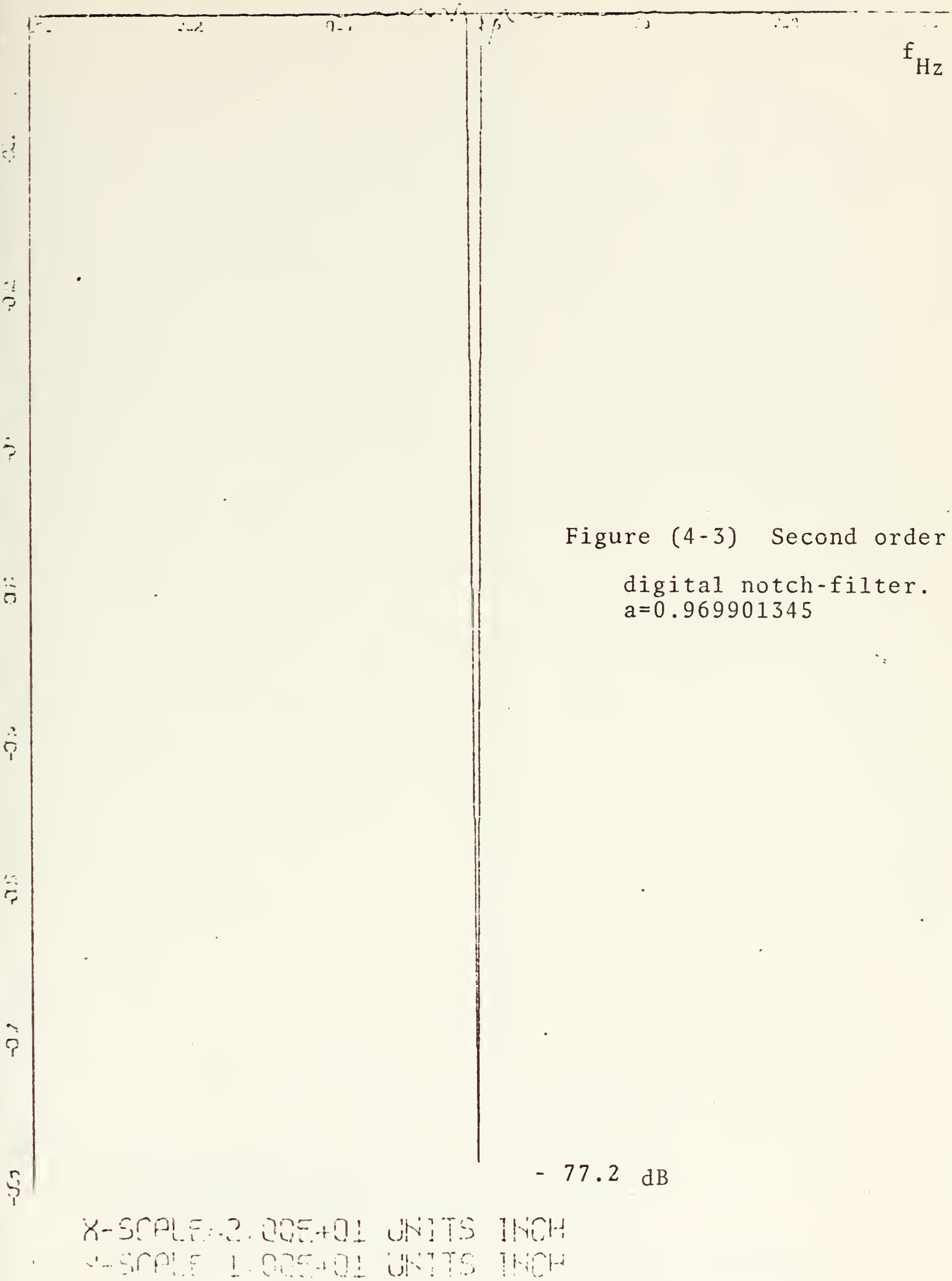
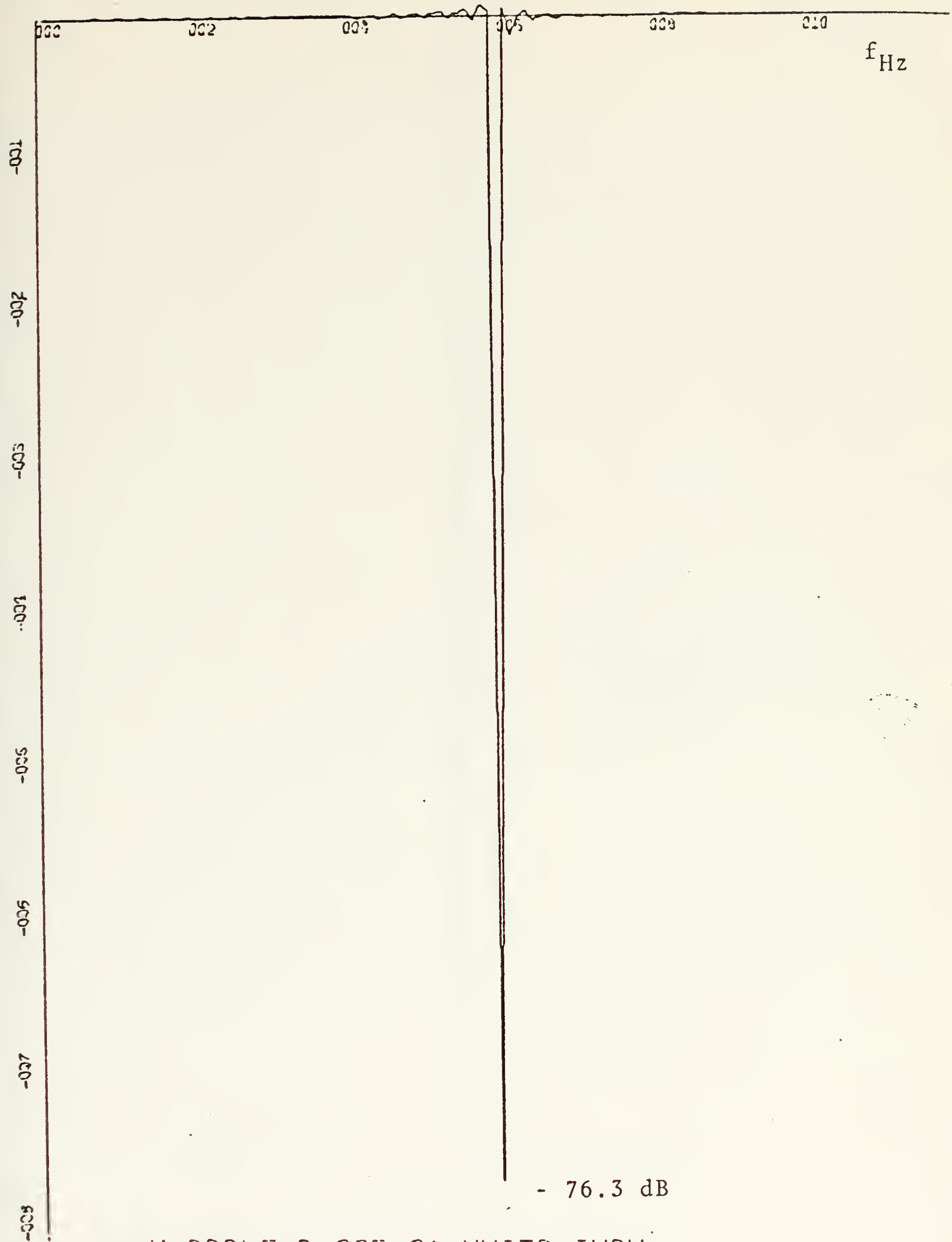


Figure (4-3) Second order digital notch-filter. a=0.969901345



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-4) Second order digital notch-filter.

a=0.979

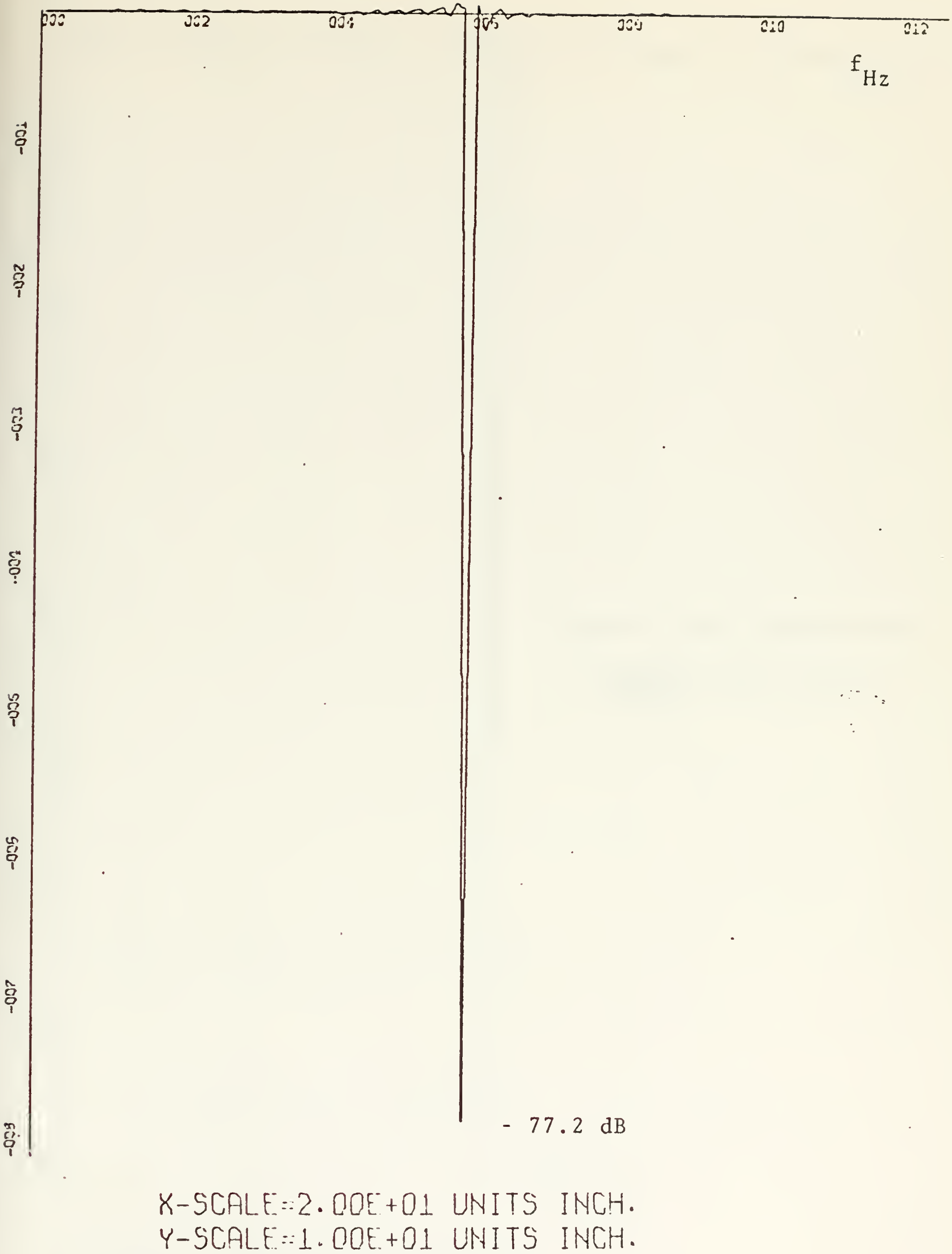


Figure (4-5) Second order digital notch-filter.
a=0.98

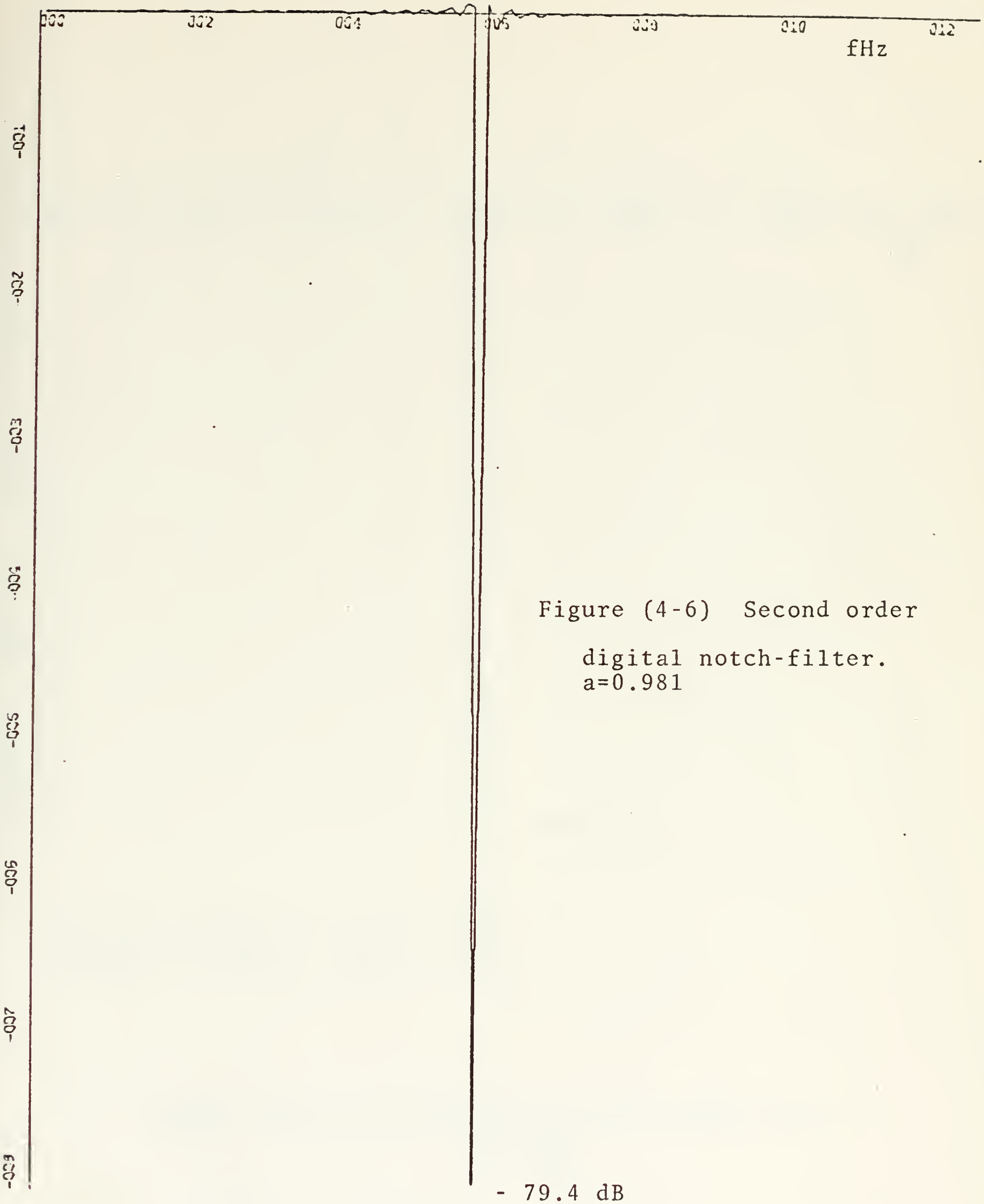
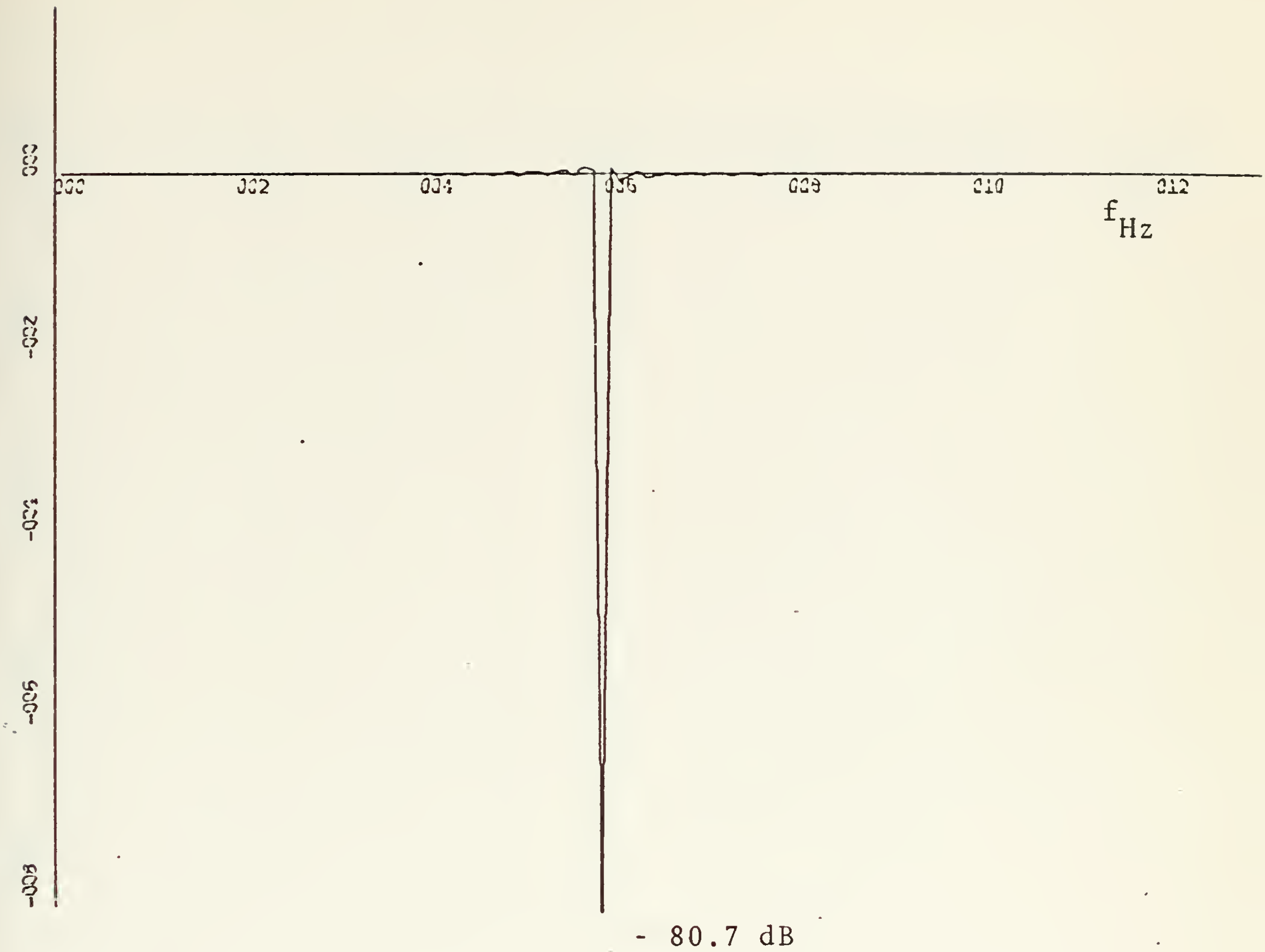


Figure (4-6) Second order digital notch-filter. a=0.981

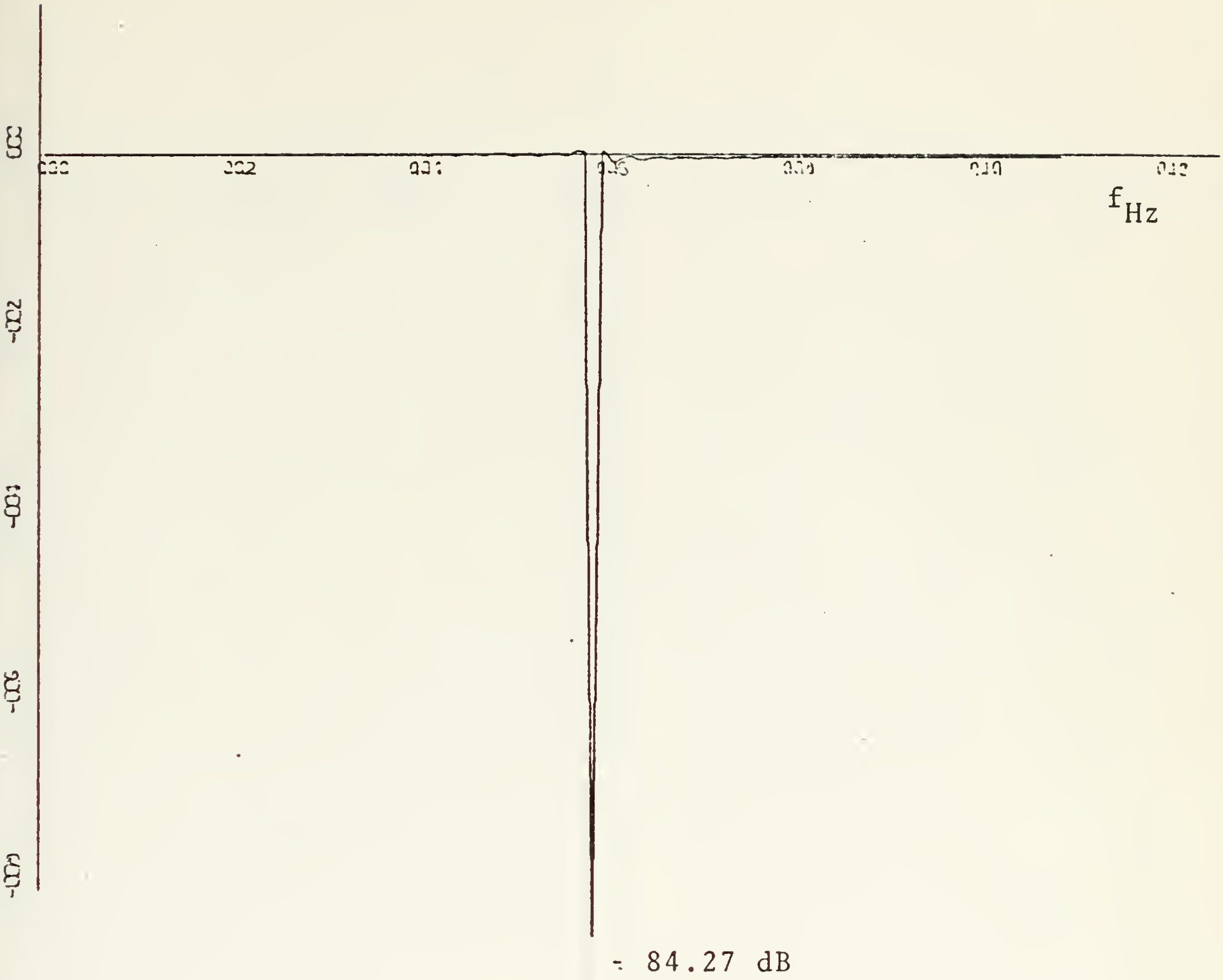
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+01 UNITS INCH.

Figure (4-7) Second order digital notch-filter.

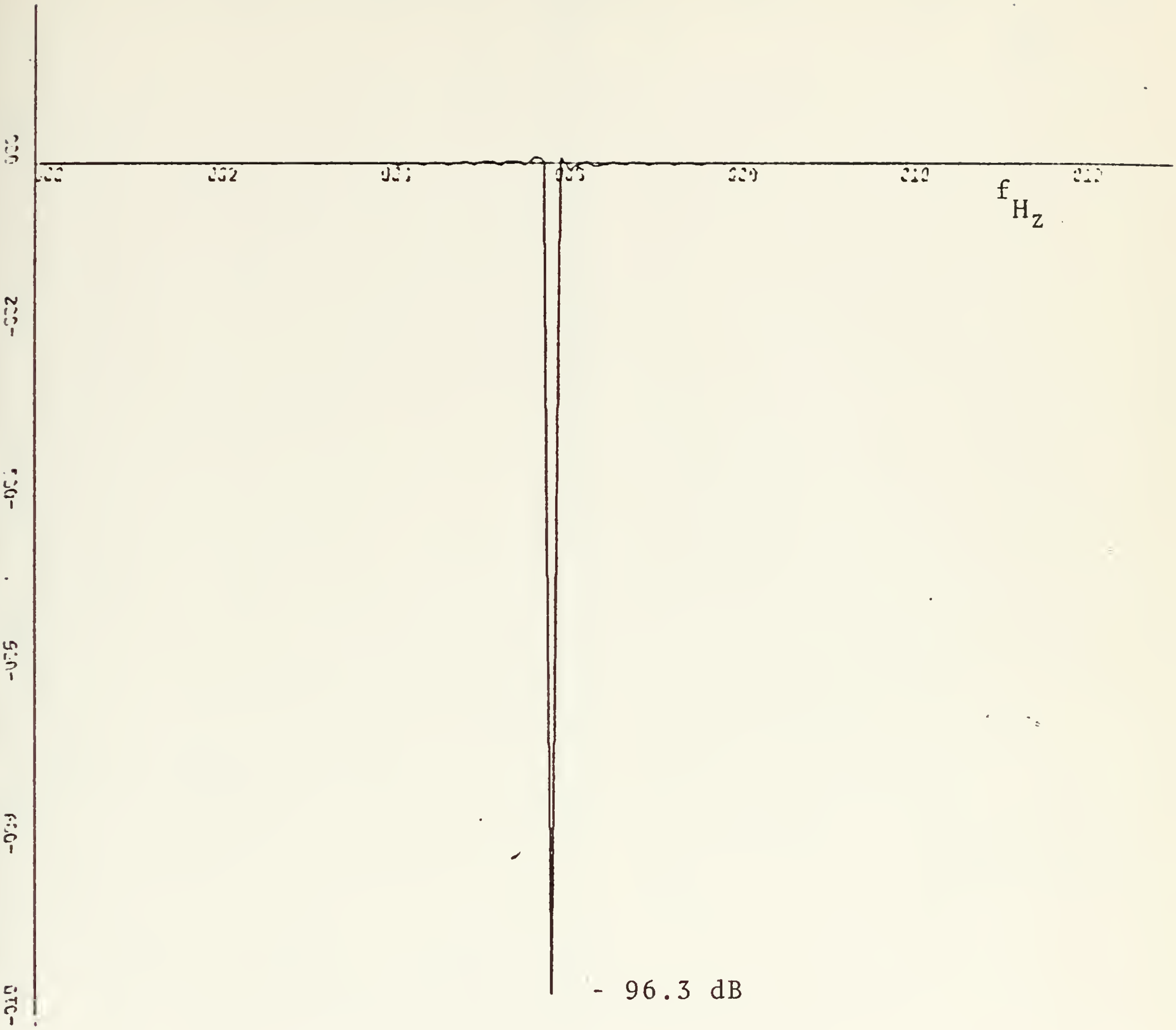
a=0.9815



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+01 UNITS INCH.

Figure (4-8) Second order digital notch-filter.

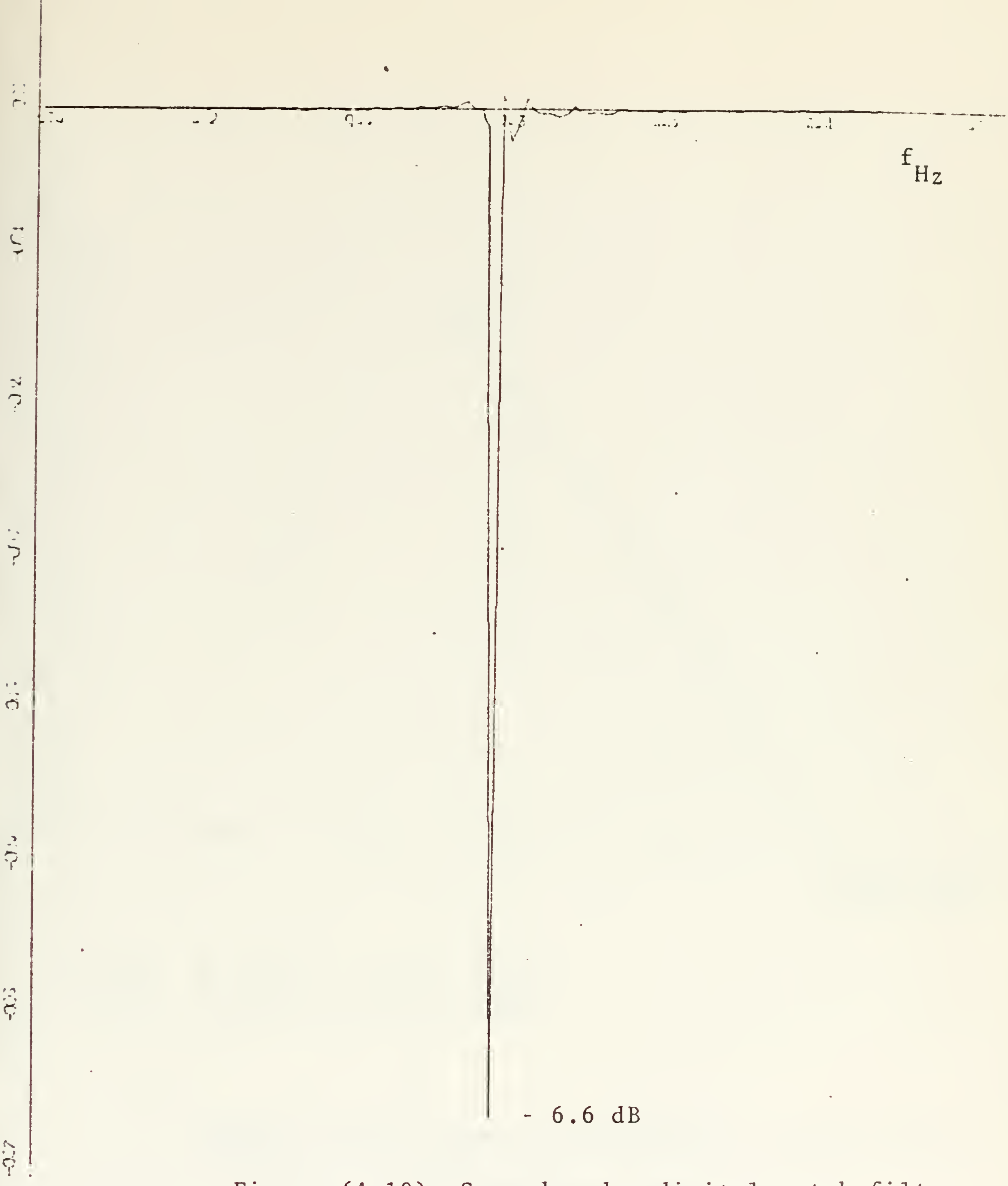
a=0.982



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+01 UNITS INCH.

Figure (4-9) Second order digital notch-filter.

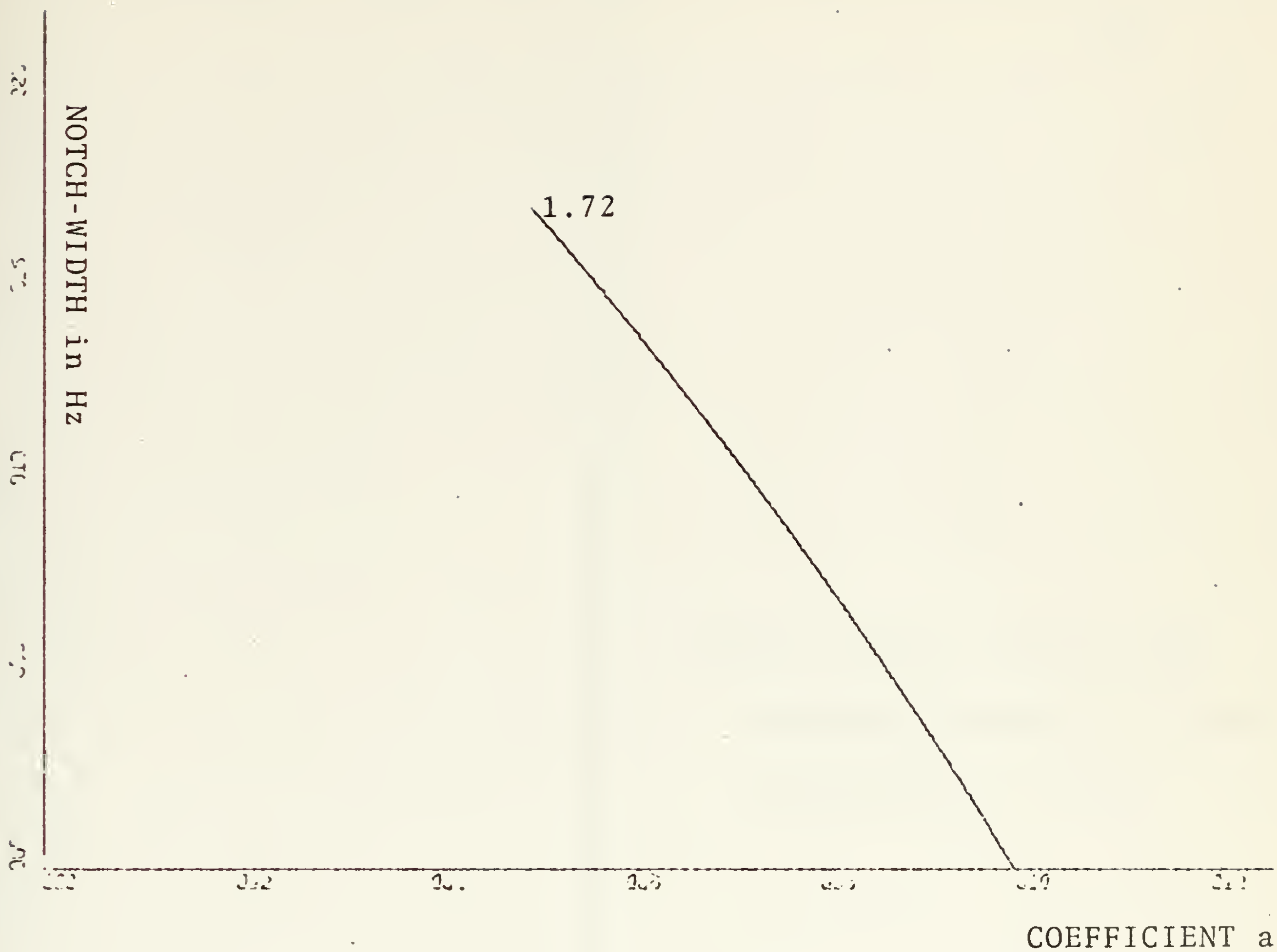
a=0.98298407



Figuer (4-10) Second order digital notch-filter.

a=0.99845153

X-SCALE 2.00E+01 UNITS INCH.
Y-SCALE 1.00E+00 UNITS INCH



X-SCALE=2.00E-01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH

Figure (4-11) Plot of notch-width vs coefficient, in theory, of equation (2-19).

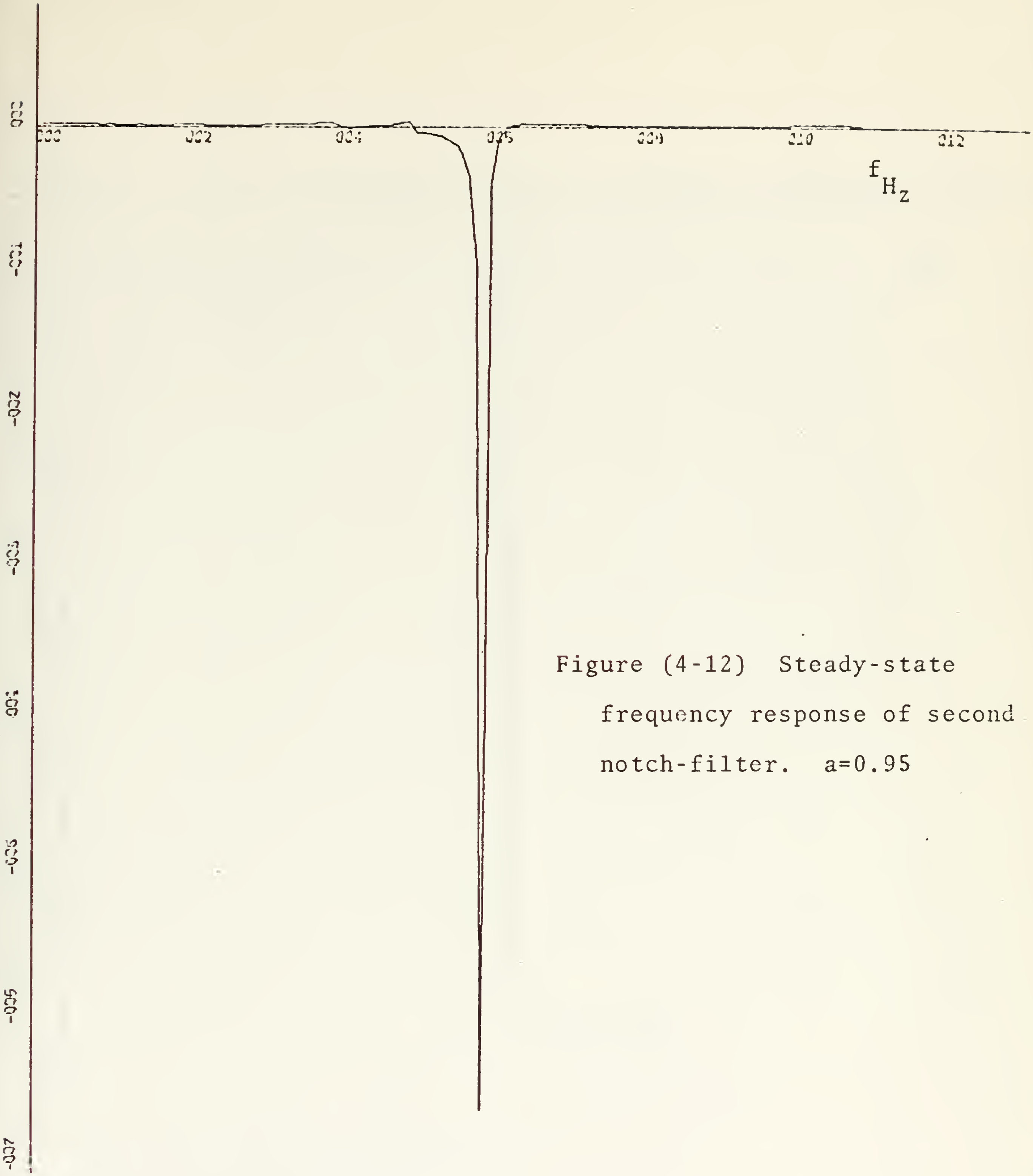
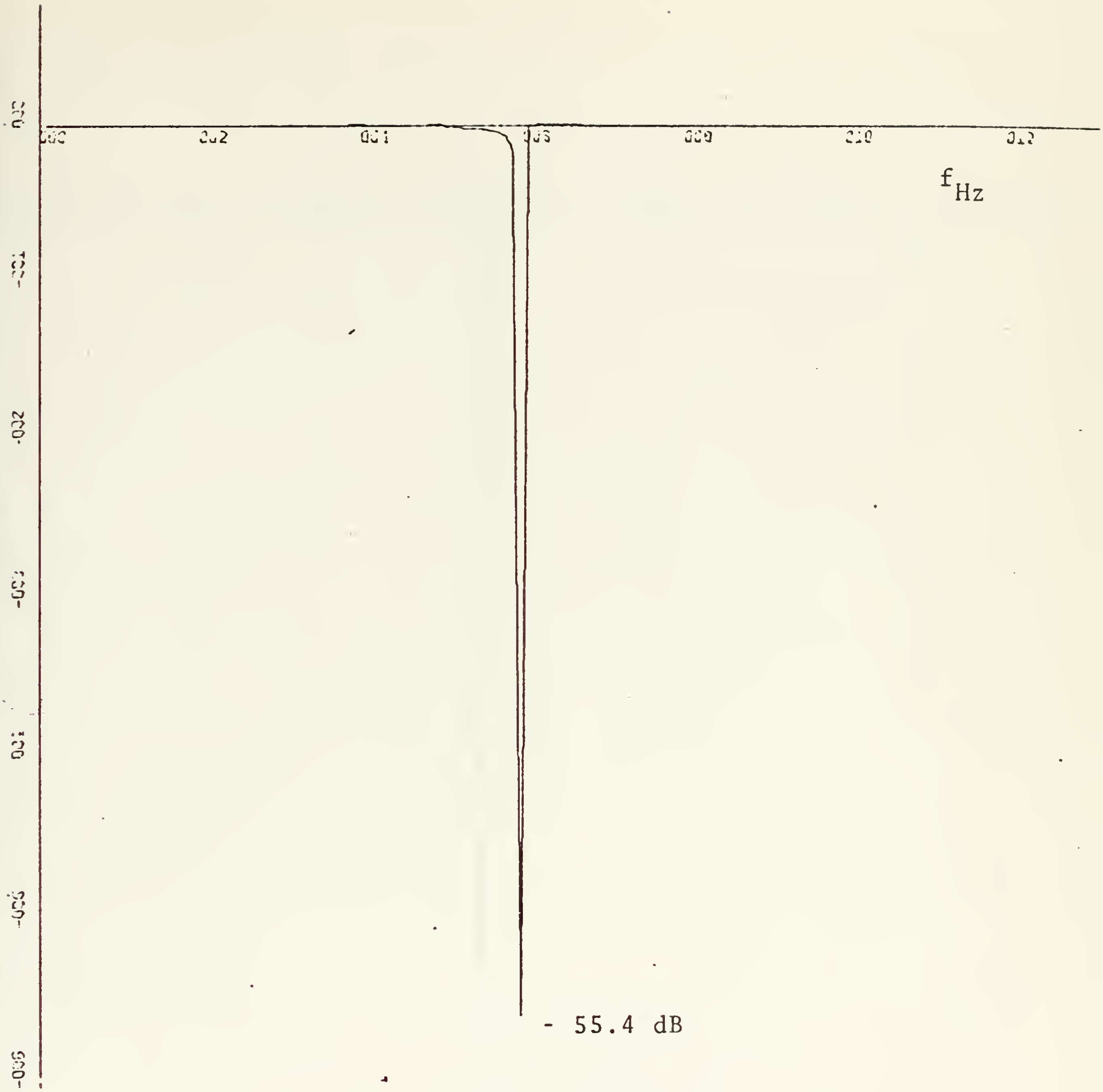


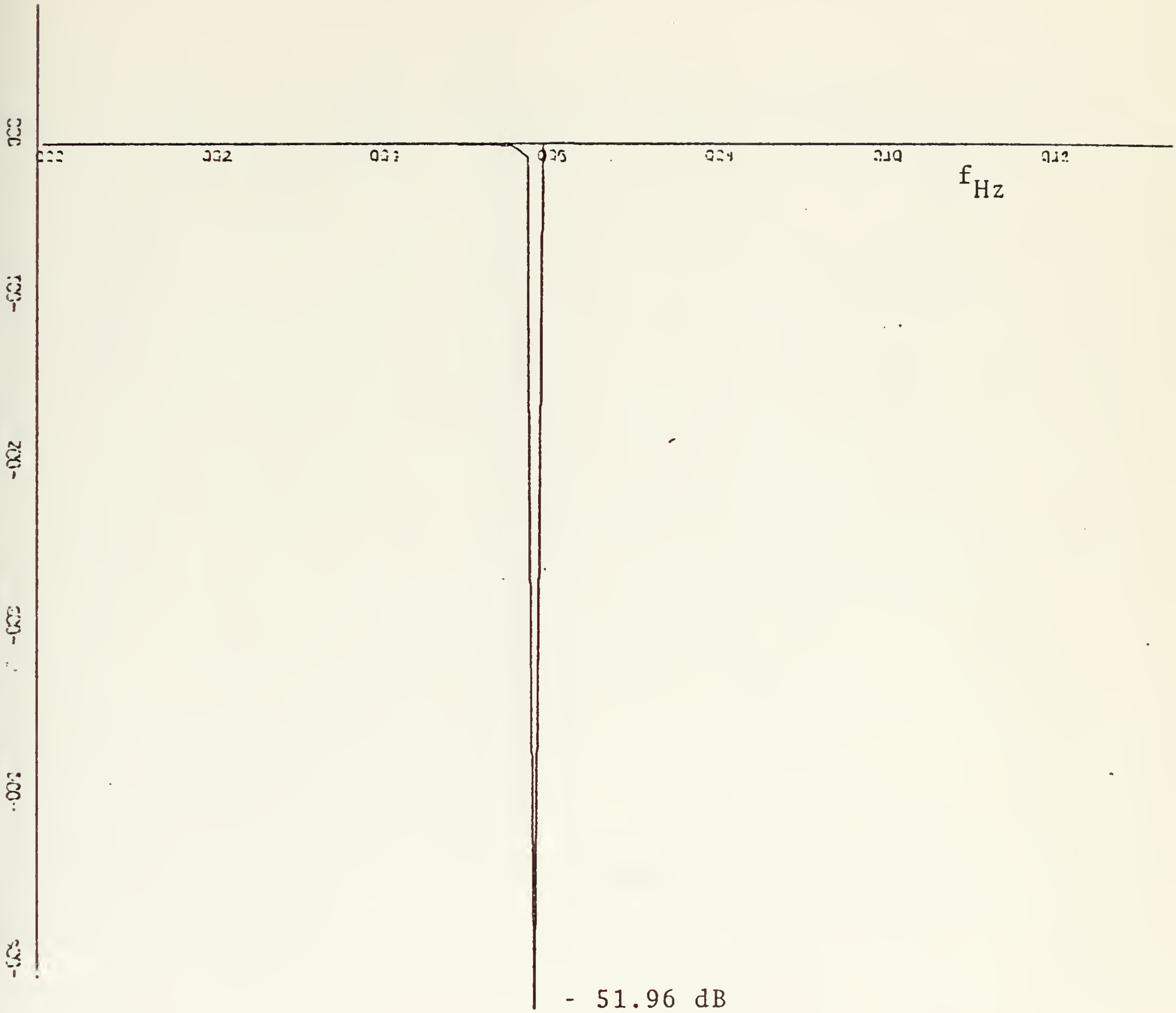
Figure (4-12) Steady-state frequency response of second notch-filter. a=0.95

X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.



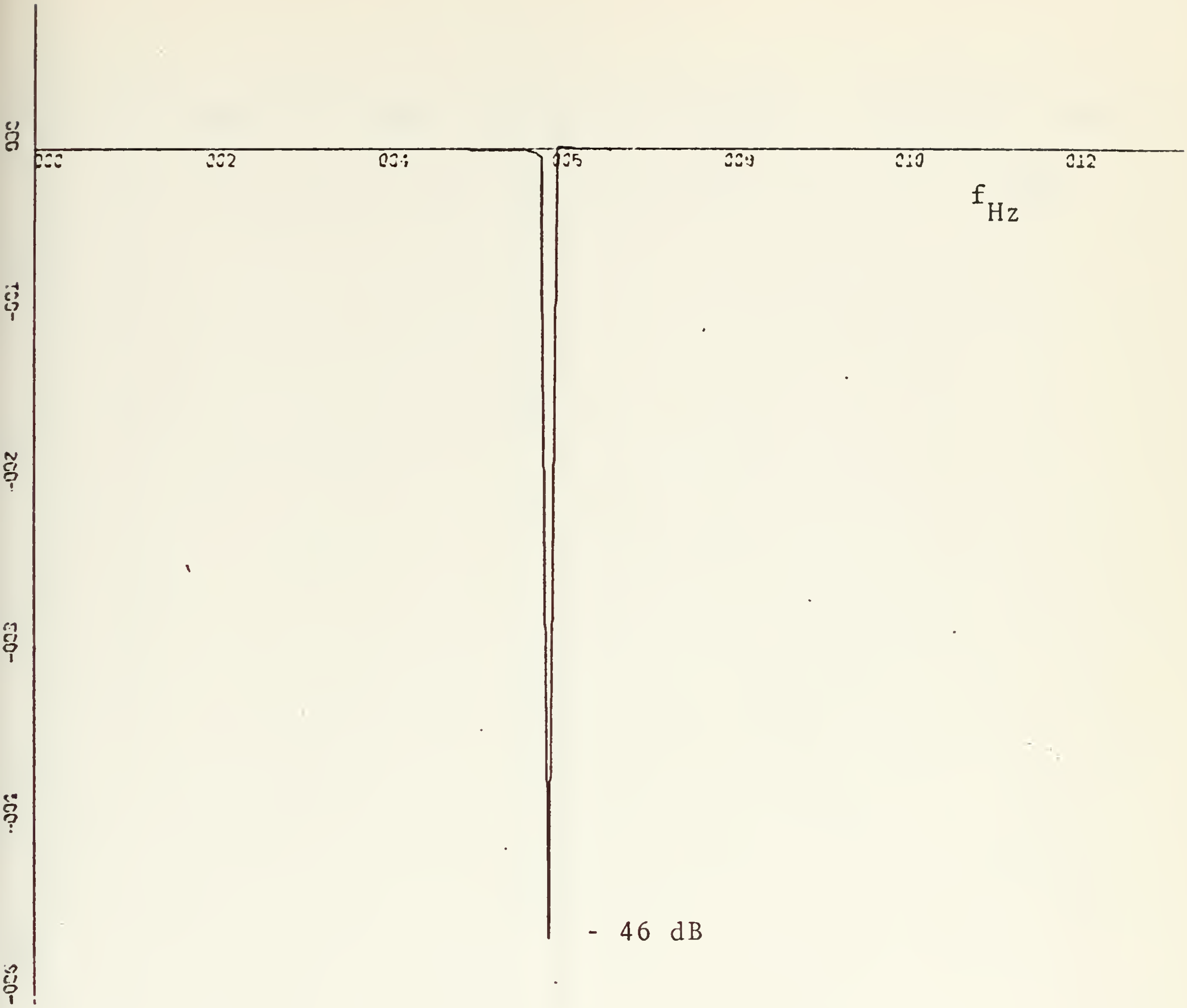
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-13) Steady-state gain frequency response of second order notch-filter with 6000 iterations. a=0.985



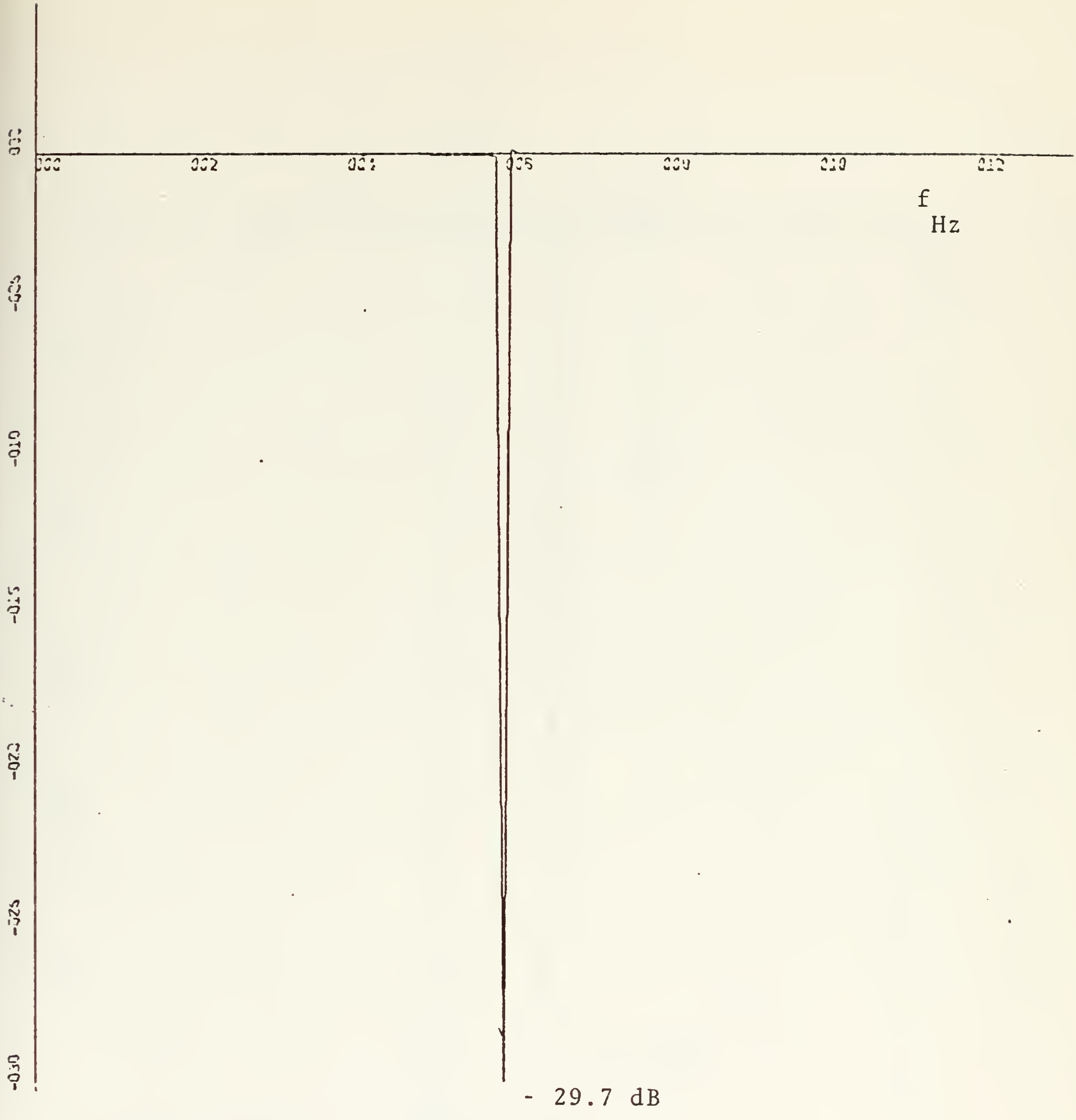
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-14) Steady-state frequency response of second order notch-filter. a=0.99



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-15) Steady-state gain frequency response of second order notch-filter.
a=0.995



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=5.00E+00 UNITS INCH.

Figure (4-16) Steady-state gain frequency response of second order notch-filter.
a=0.999

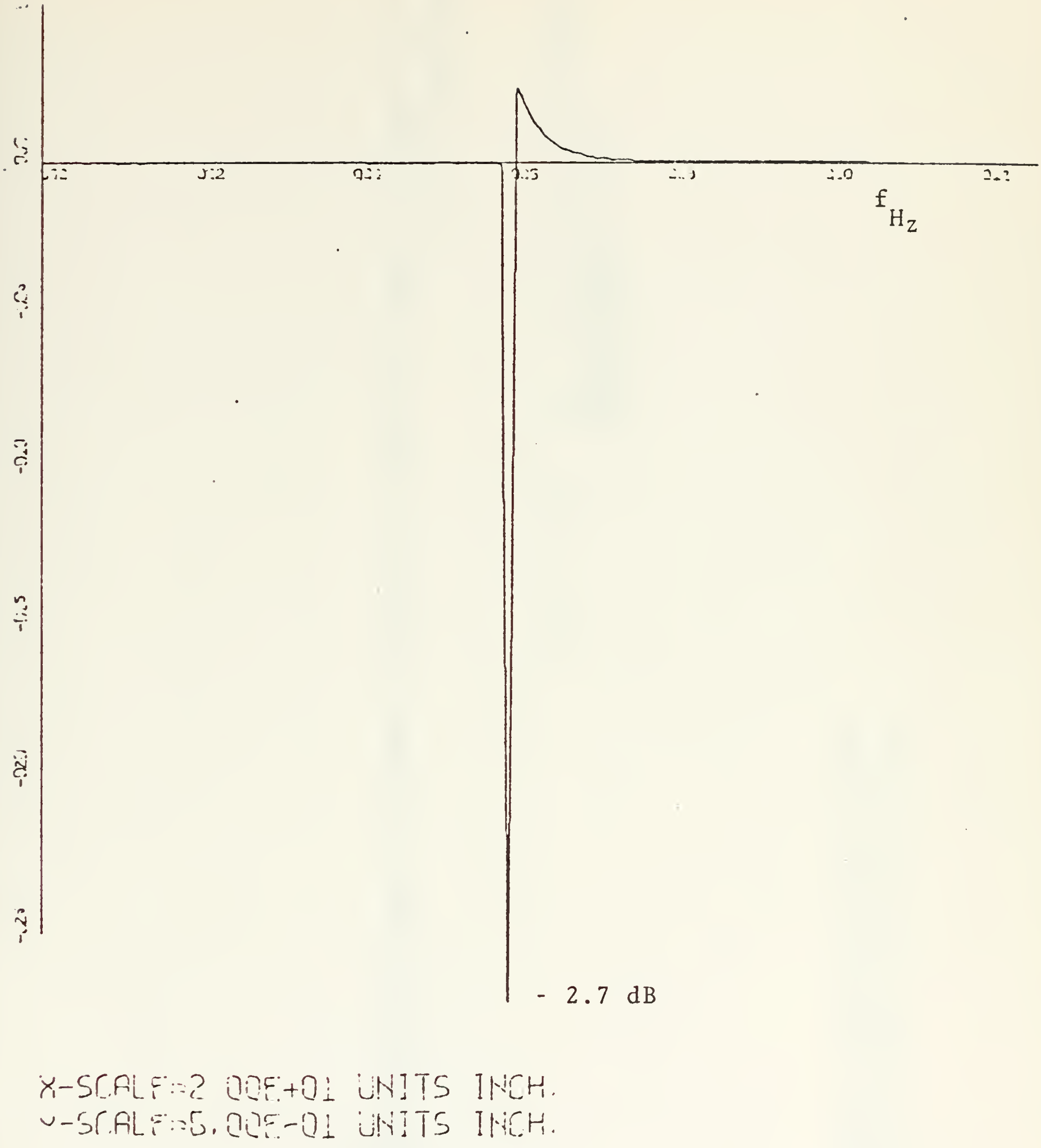


Figure (4-17) Steady-state gain frequency response of second order notch-filter.
a=0.9999

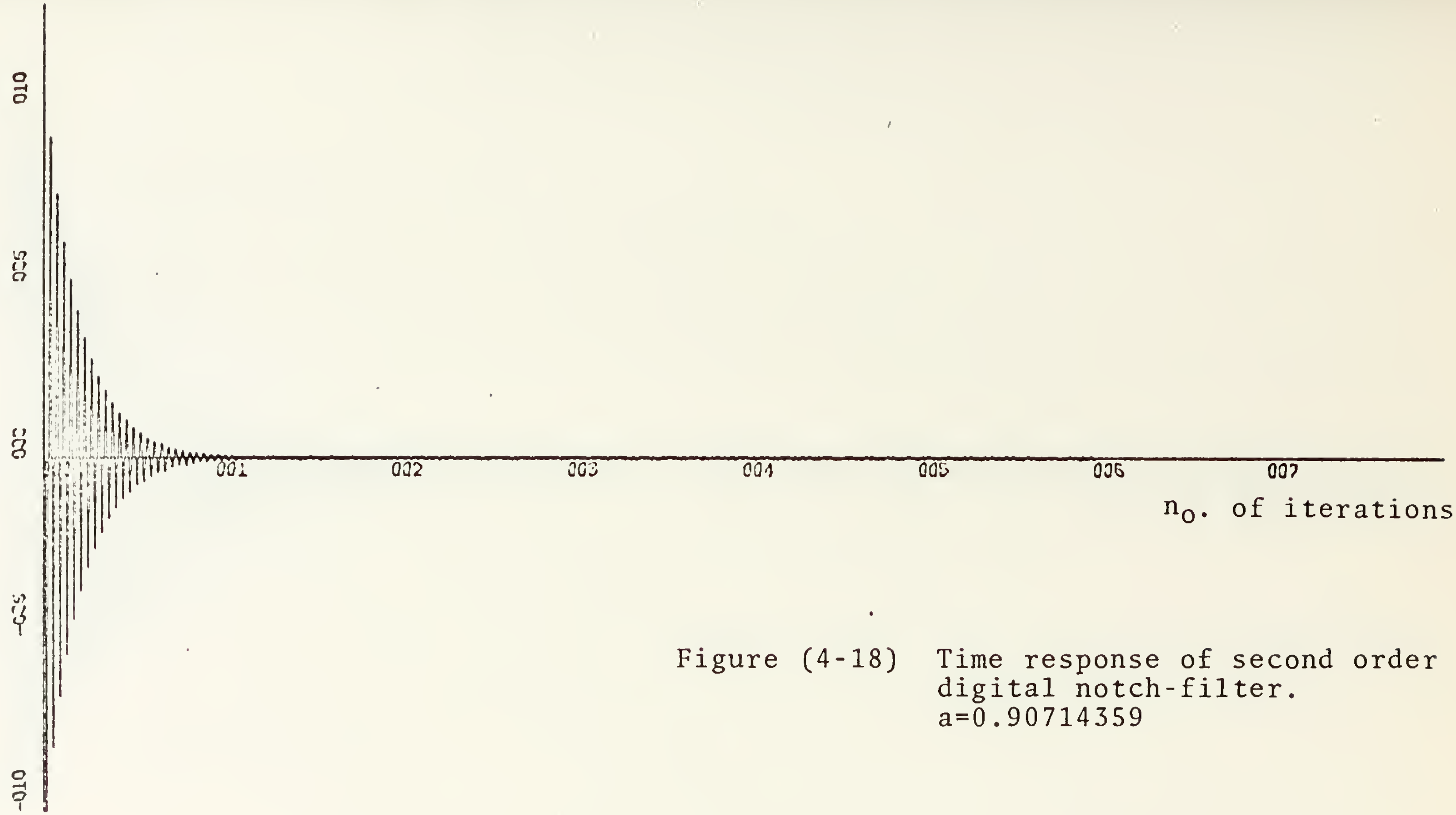


Figure (4-18) Time response of second order digital notch-filter.
a=0.90714359

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

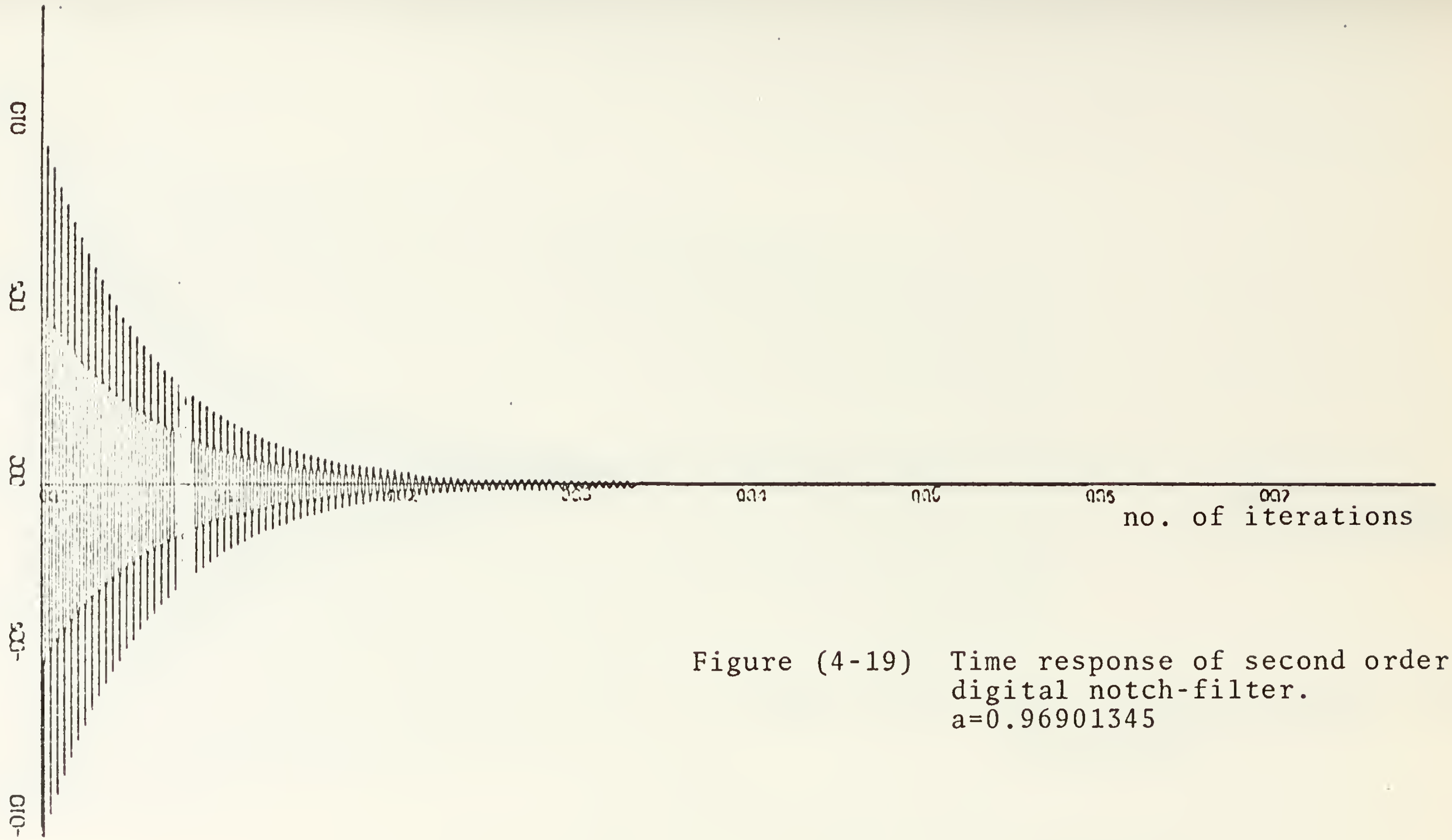


Figure (4-19) Time response of second order digital notch-filter. a=0.96901345

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

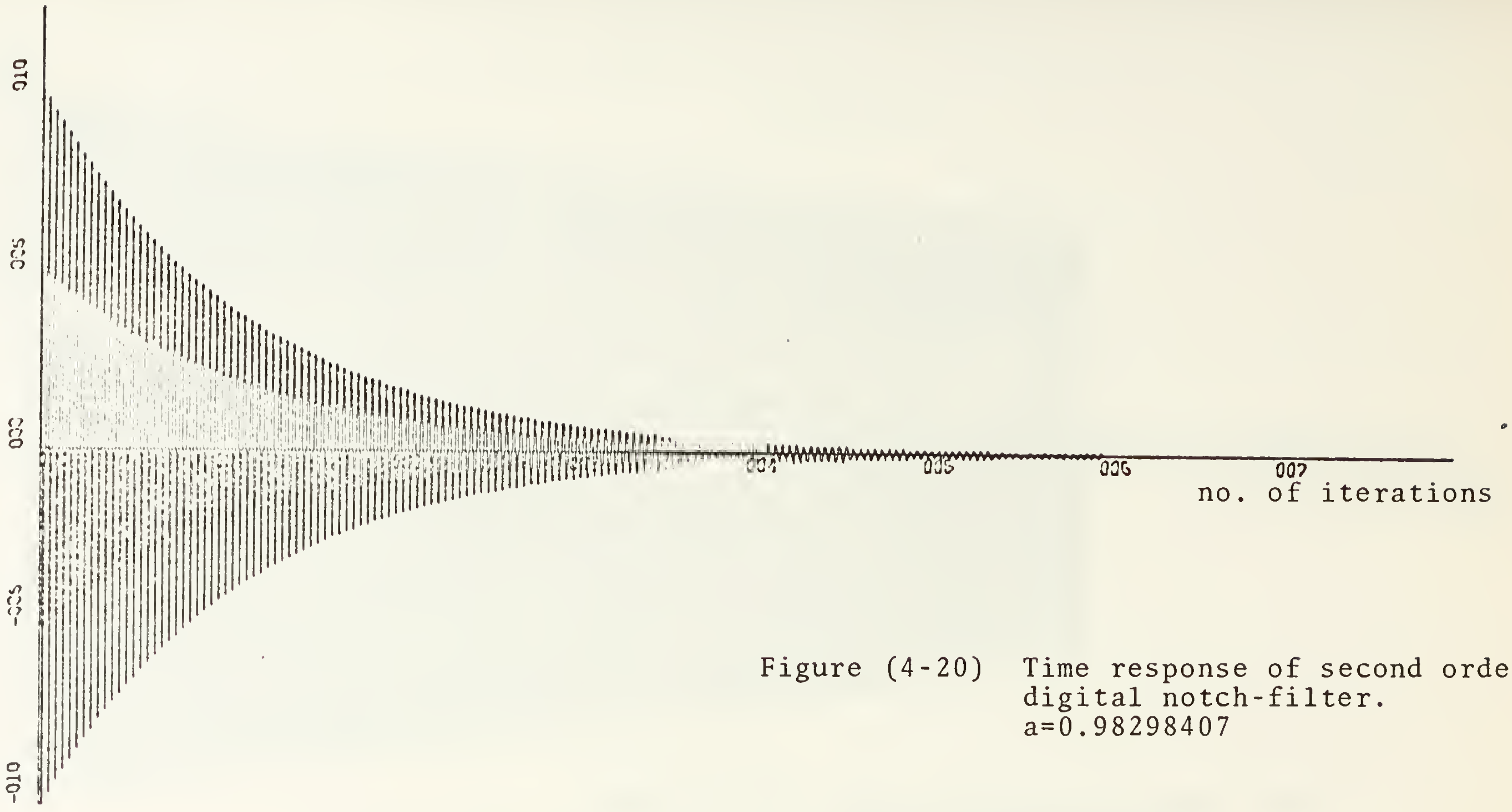


Figure (4-20) Time response of second order digital notch-filter.
a=0.98298407

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

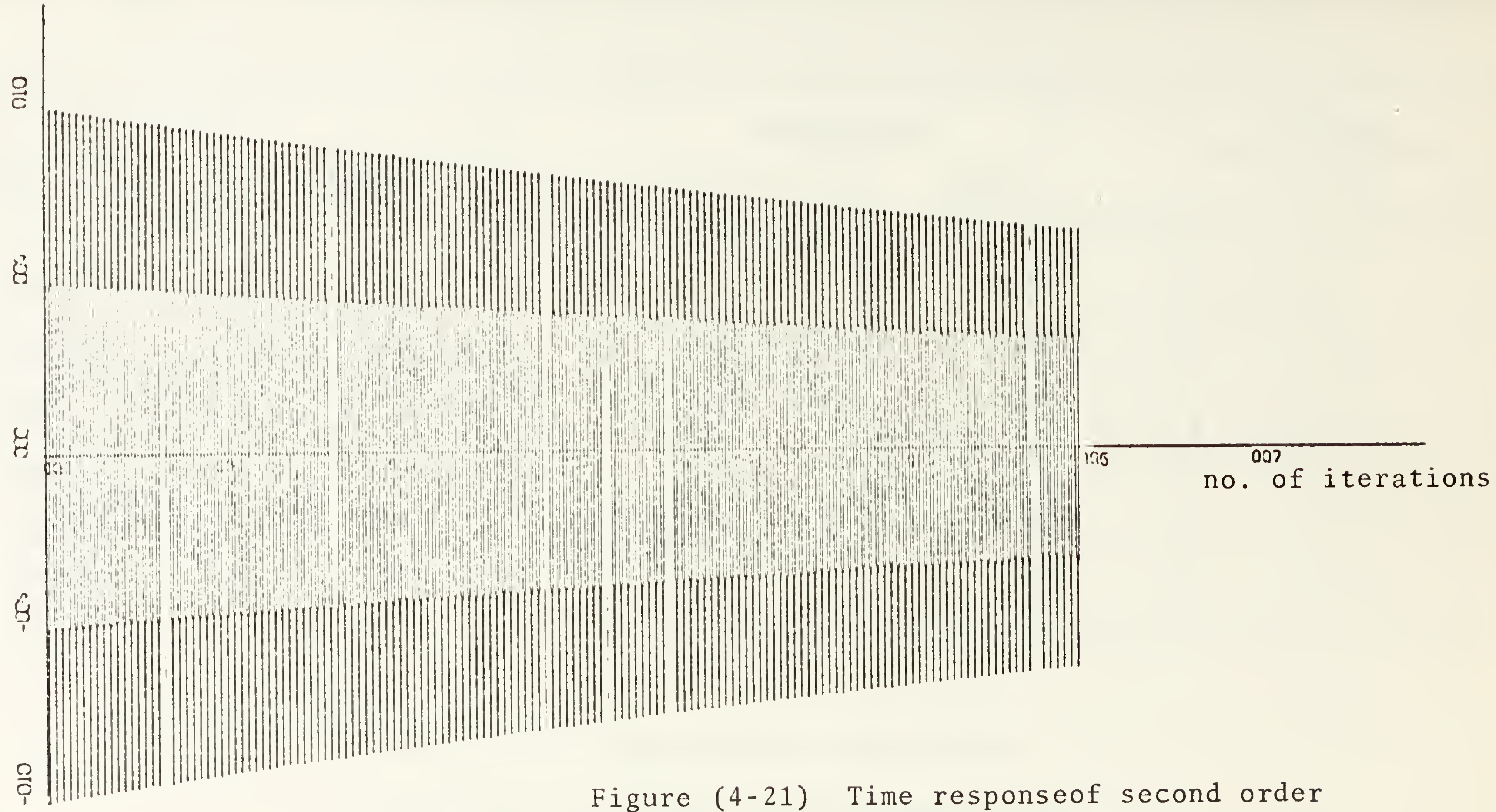


Figure (4-21) Time responseof second order digital notch-filter. a=0.99845153

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

COEFFICIENT	NOTCH-GAIN in dB	NOTCH-WIDTH in Hz		MAXIMUM PASSBAND RIPPLE in dB
		PRACTICE	THEORY	
0.95	- 65.6	3	0.2	1.67
0.985	- 55	< 1	0.06	0.26
0.99	- 52	< 1	0.04	0.18
0.995	- 46	< 1	0.02	0.1
0.999	- 29.7	< 1	0.007	0.25
0.9999	- 2.7	< 1	0.006	0.25

Table (4-1) Second order notch-filter.

The results are based on the steady-state frequency response of 6000 iterations.

It is noted that the passband ripple is large for frequencies very close to the notch-frequency. Further, about one hertz from the notch, all passband starts less than 0.09 dB in magnitude and smoother for going farther the notch.

2. Third Order Digital Notch-Filter

The third order digital notch-filter is characterized by the transfer function

$$H(z) = \frac{1 + z^{-3}}{1 + az^{-3}} \qquad (4\text{-}4)$$

and realized as follows

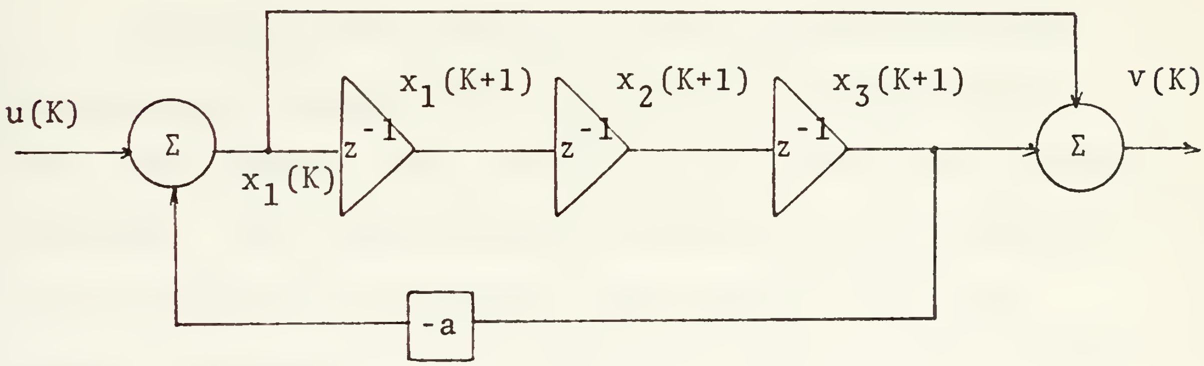


Figure (4-22) Canonic realization of third-order digital notch-filter.

The corresponding state equations are derived as

$$
\begin{aligned}
x_1\ (K) &= -\ a\ x_2\ (K+1) + u\ (K) \\
x_1\ (K+1) &= x_1\ (K) \\
x_2\ (K+1) &= x_1\ (K+1) \qquad\qquad (4\text{-}5) \\
x_3\ (K+1) &= x_2\ (K+1) \\
v\ (K) &= x_3\ (K+1) + x_1\ (K)
\end{aligned}
$$

and the sampling time required for 60 Hz-notch is given as

$$T = \frac{1}{2mf_0} = \frac{1}{2 \times 3 \times 60} = 2.77777778 \times 10^{-3} \text{ sec.}$$

Similar to the results of the second order case, the frequency responses of a third order digital notch-filter are shown in Figs. (4-23) to (4-31) with one thousand iterations. The steady state of third order is reached in about ten thousand iterations. The steady-state gain frequency responses are illustrated in Figs. (4-32) through (4-37).

The time responses of third order digital notch-filter are shown in Figs. (4-38) to (4-41). Similarly, Table (4-1) of second order, Table (4-2) of third order is tabulated.

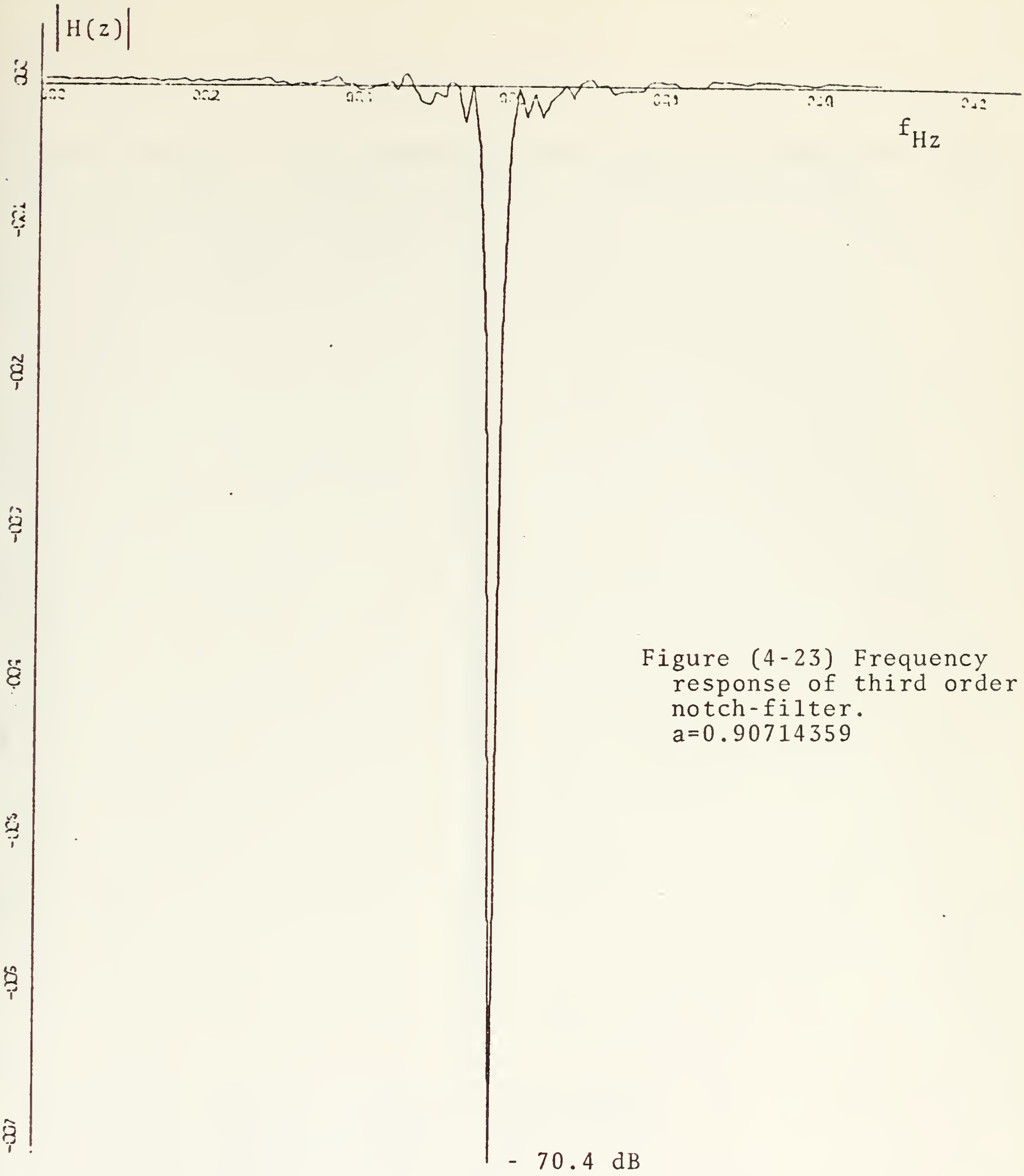
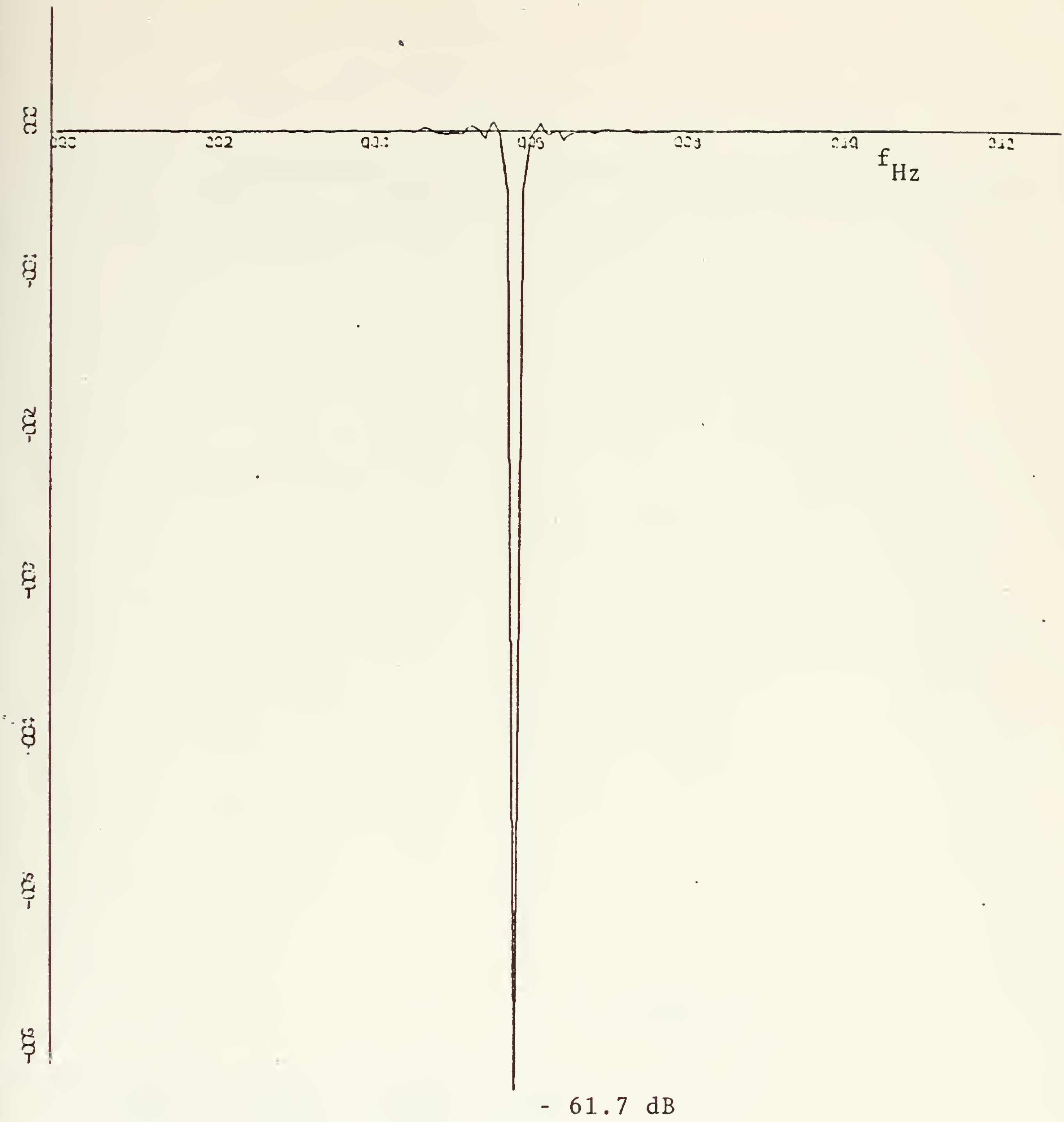


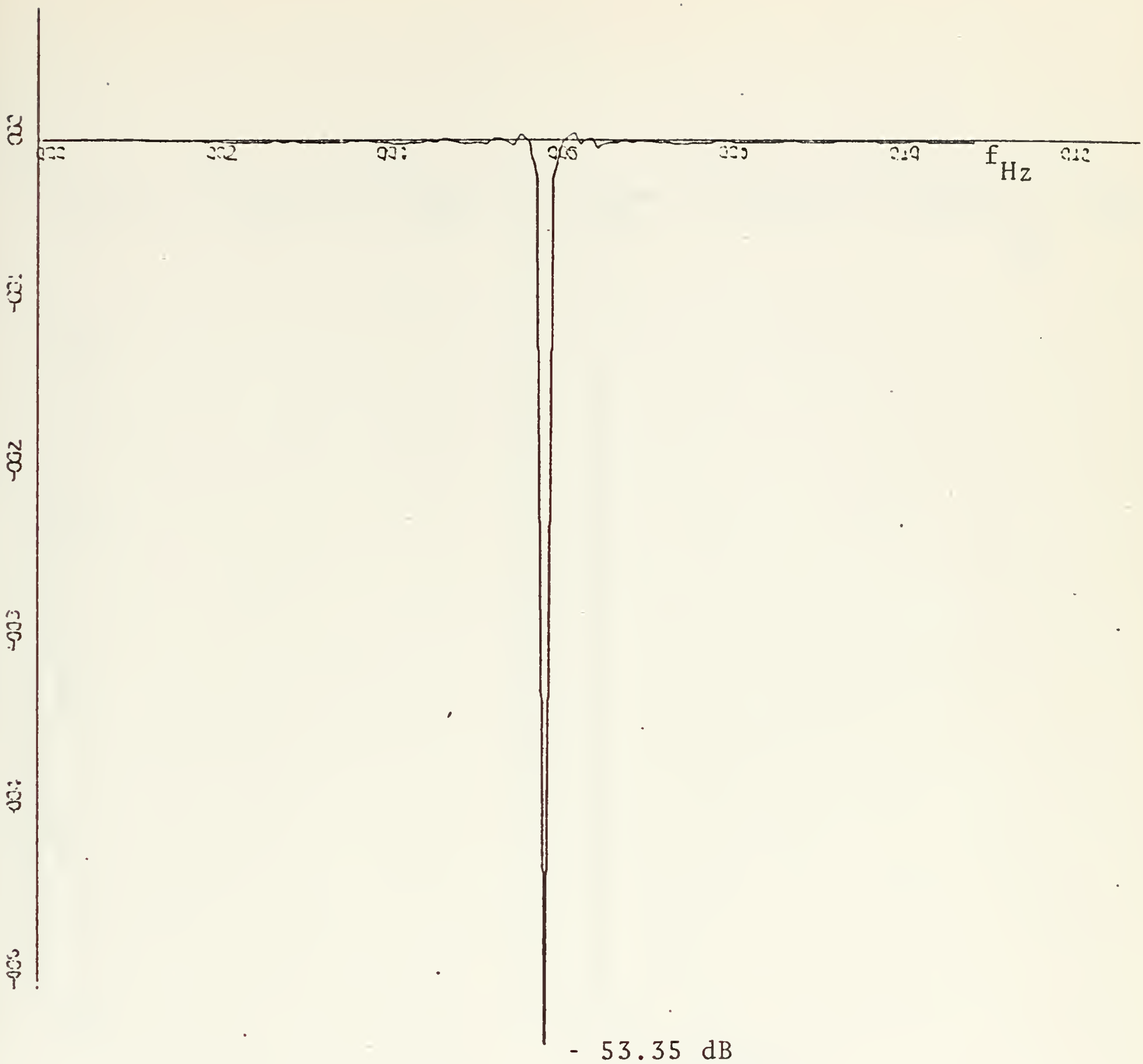
Figure (4-23) Frequency response of third order notch-filter. a=0.90714359

X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.



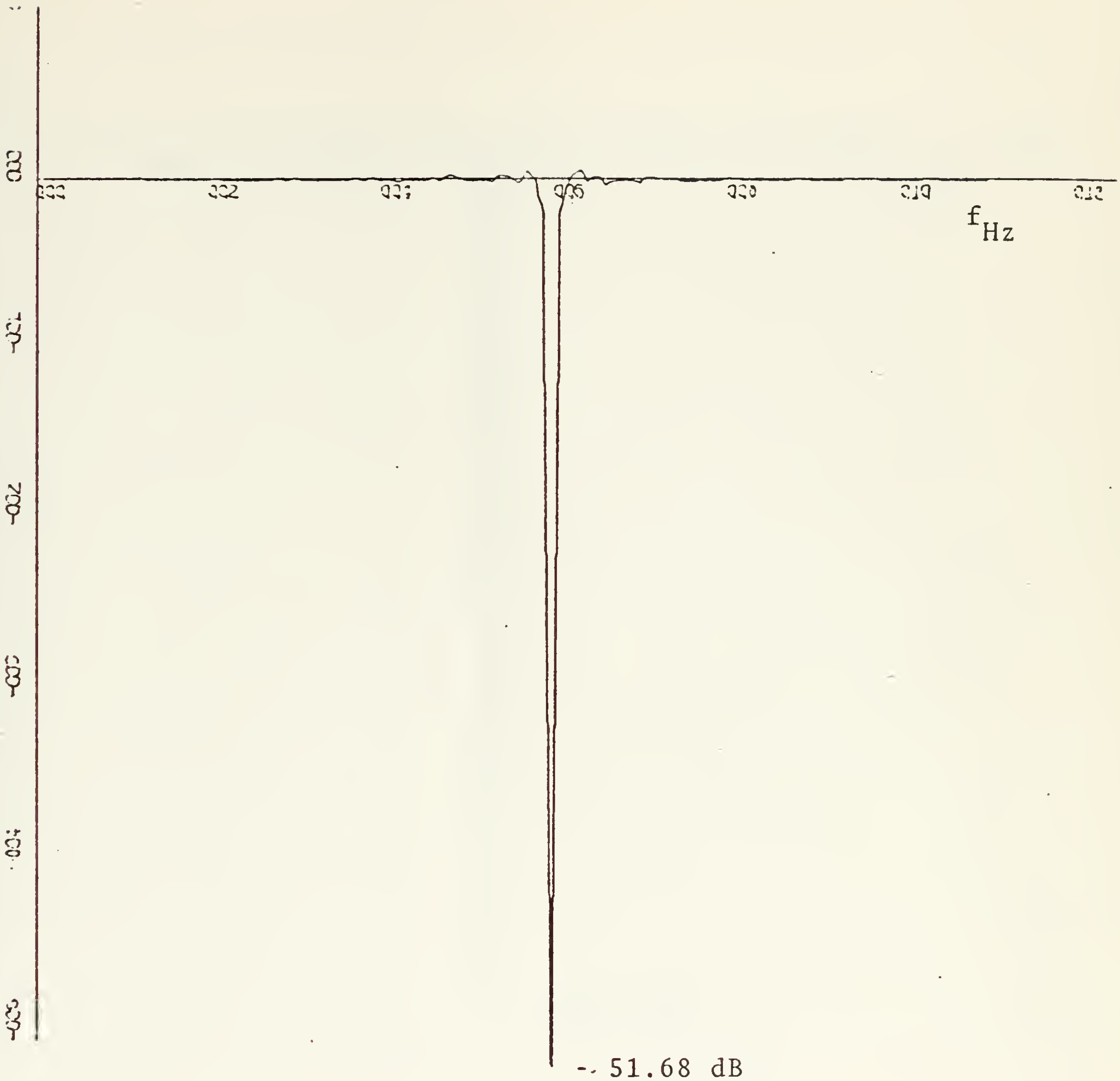
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-24) Frequency response of third order notch-filtre.
a=0.96901345



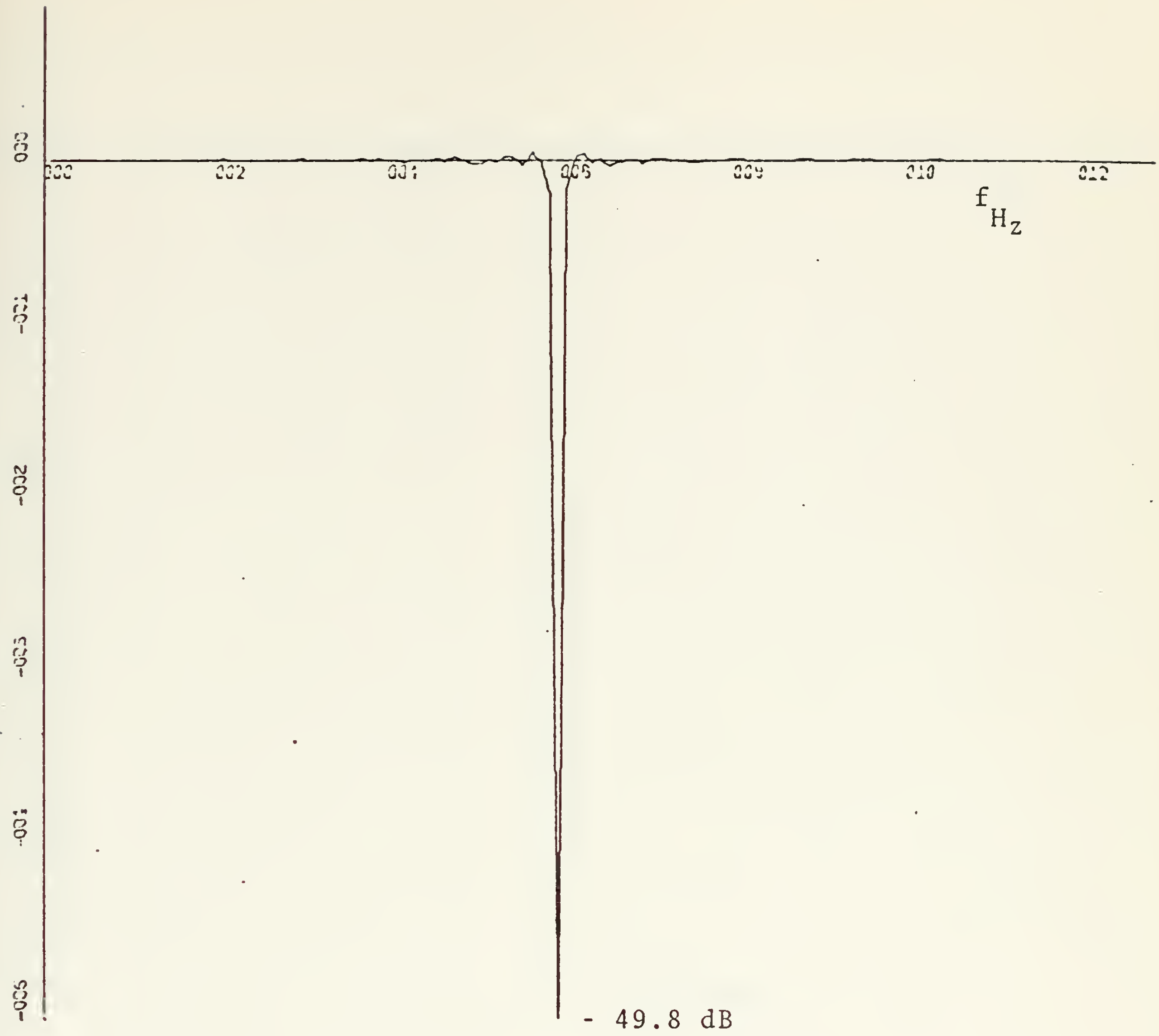
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-25) Frequency response of third order notch-filter.
a=0.979



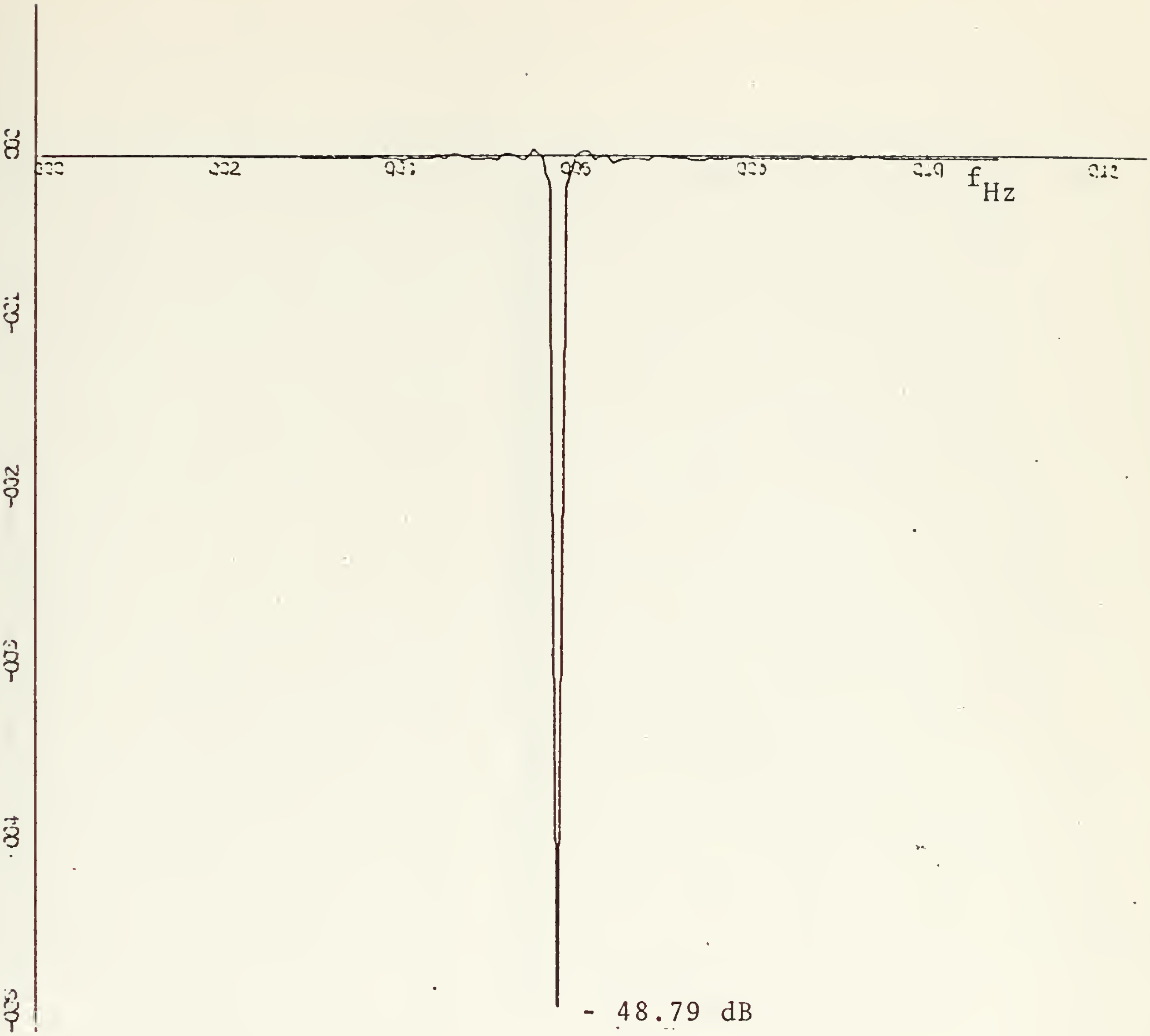
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-26) Frequency response of third order notch-filter.
a=0.98



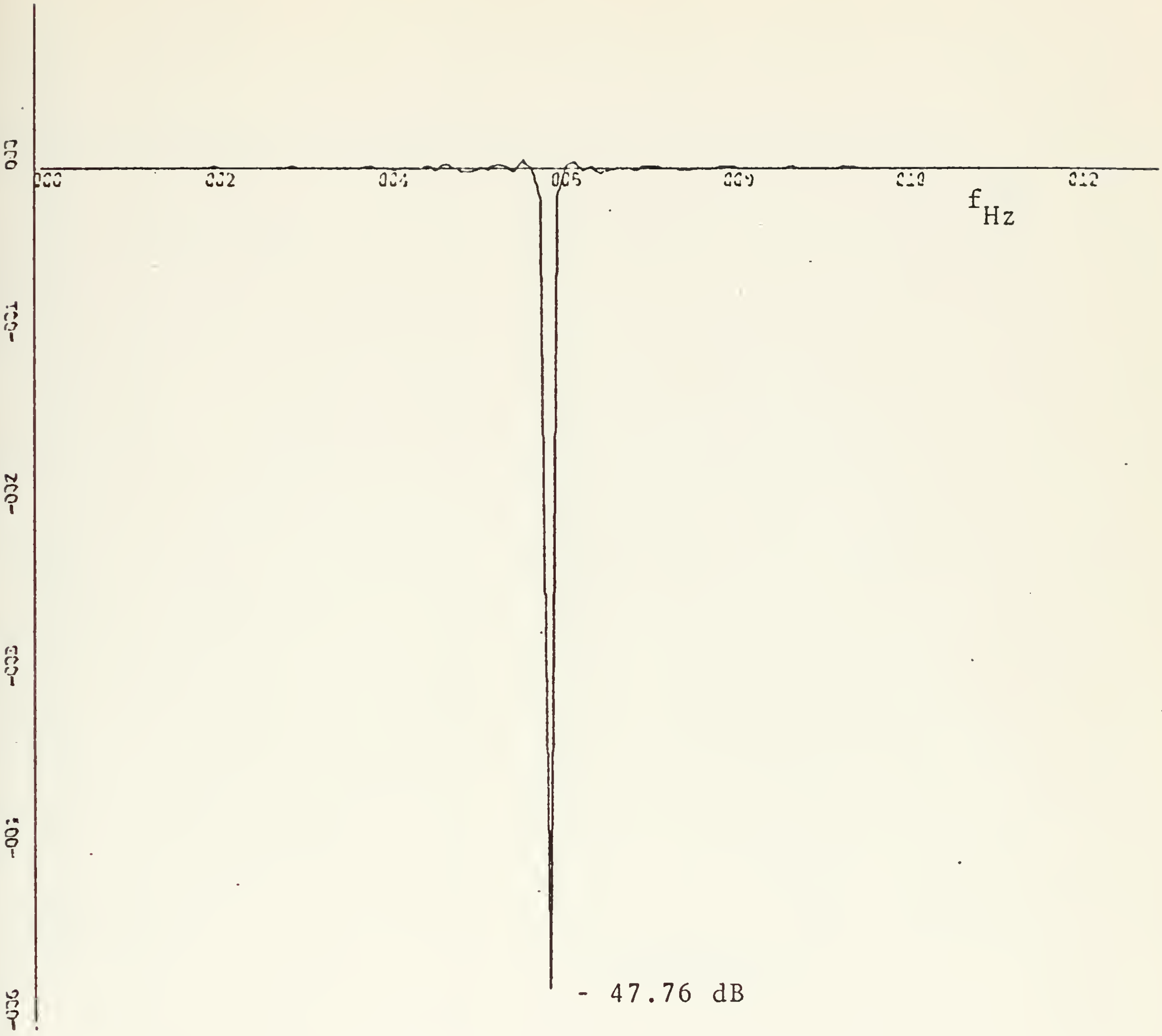
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-27) Frequency response of third order notch-filter.
a=0.981



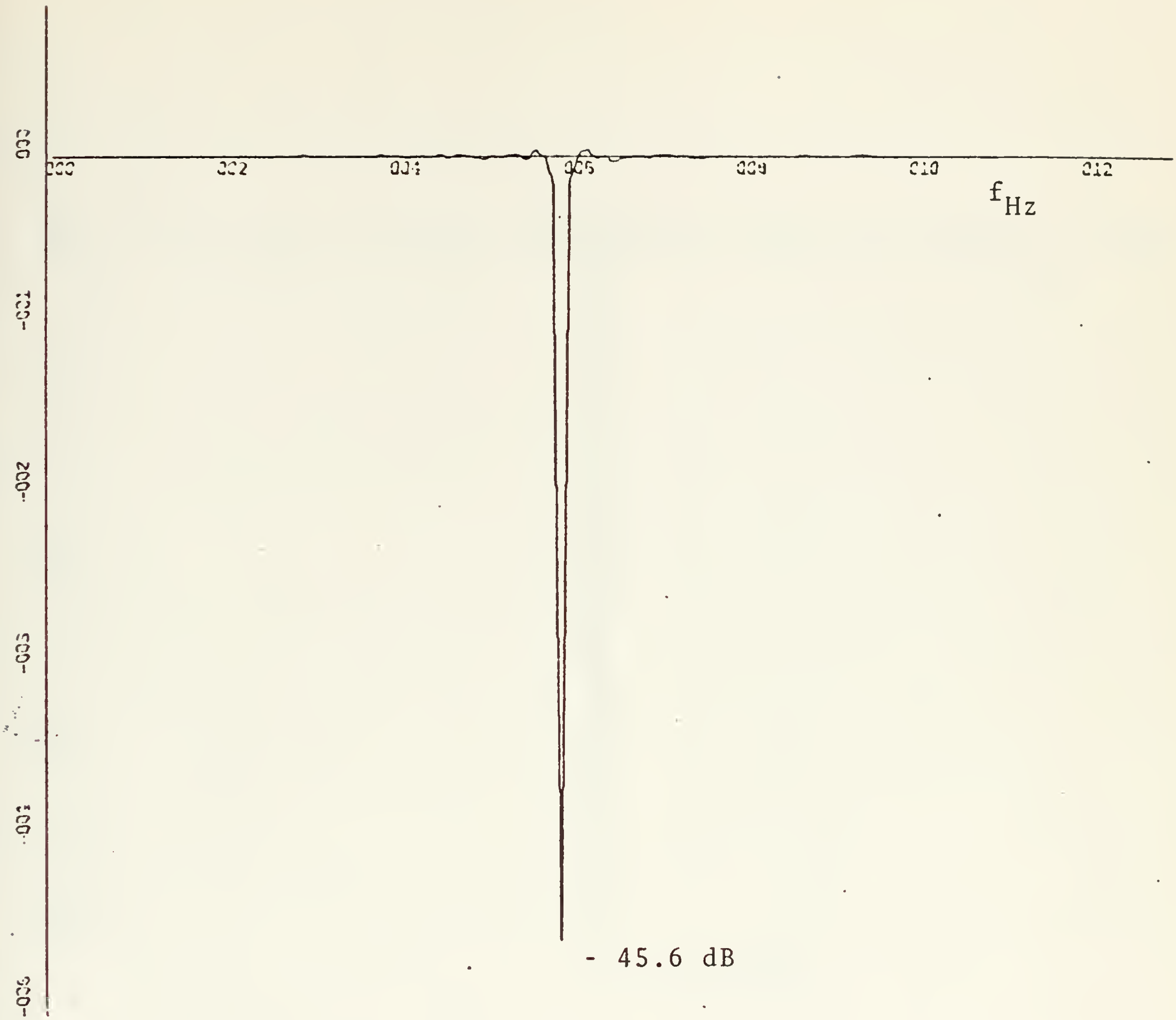
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-28) Frequency response of third order notch-filter.
a=0.9815



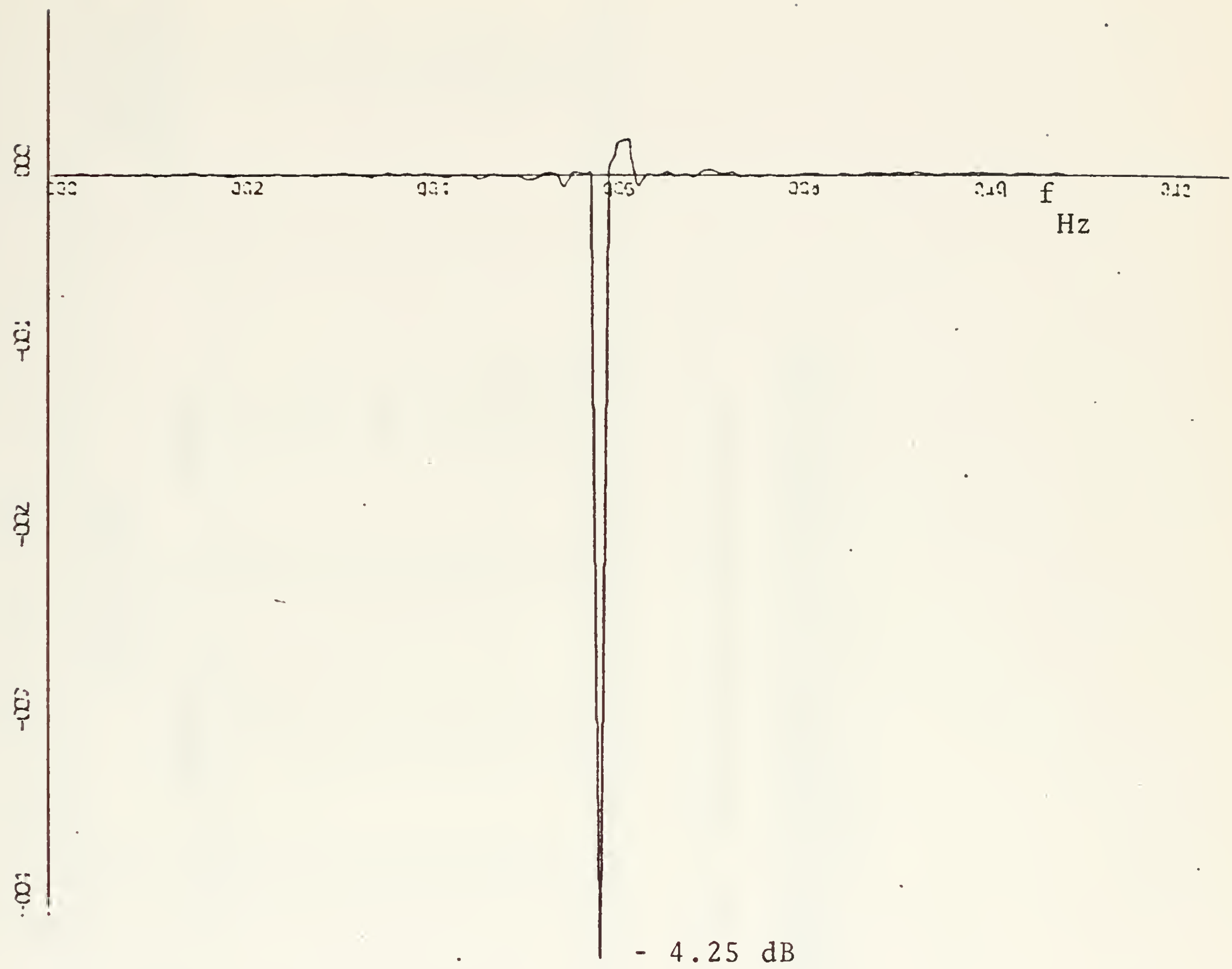
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-29) Frequency response of third order notch-filter.
a=0.982



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-30) Frequency response of third order notch-filter.
a=0.98298407



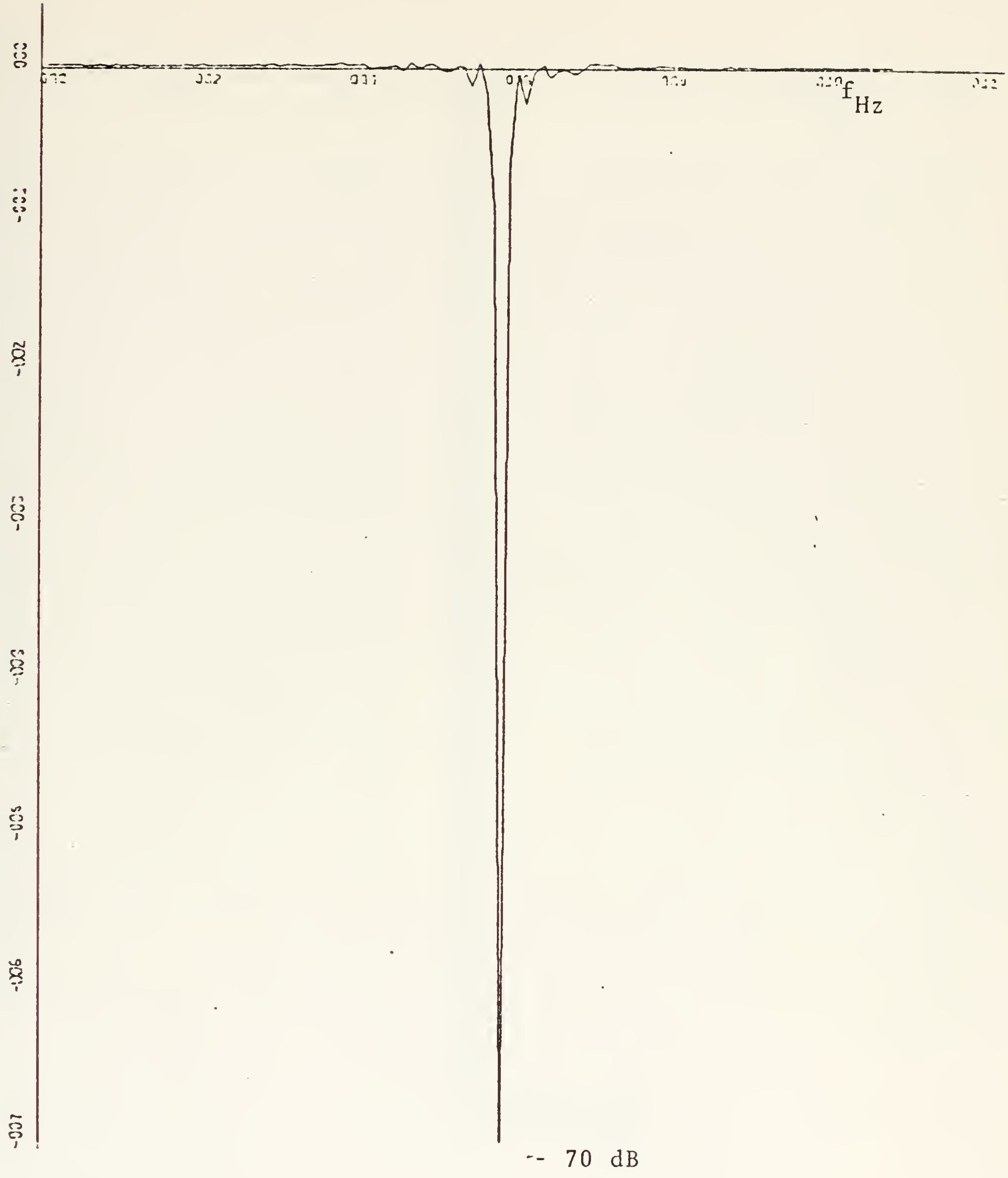
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.

Figure (4-31) Frequency response of third order notch-filter.
a=0.99845153

COEFFICIENT	NOTCH-GAIN in dB	NOTCH-WIDTH in Hz		MAXIMUM PASSBAND RIPPLE in dB
		PRACTICE	THEORY	
0.95	- 69.6	2	0.2	2.22
0.985	- 67	< 1	0.06	0.29
0.99	- 60	< 1	0.04	0.24
0.995	- 51.25	< 1	0.02	0.14
0.999	- 26	< 1	0.007	0.05
0.9999	- 2.8	< 1	0.006	0.13

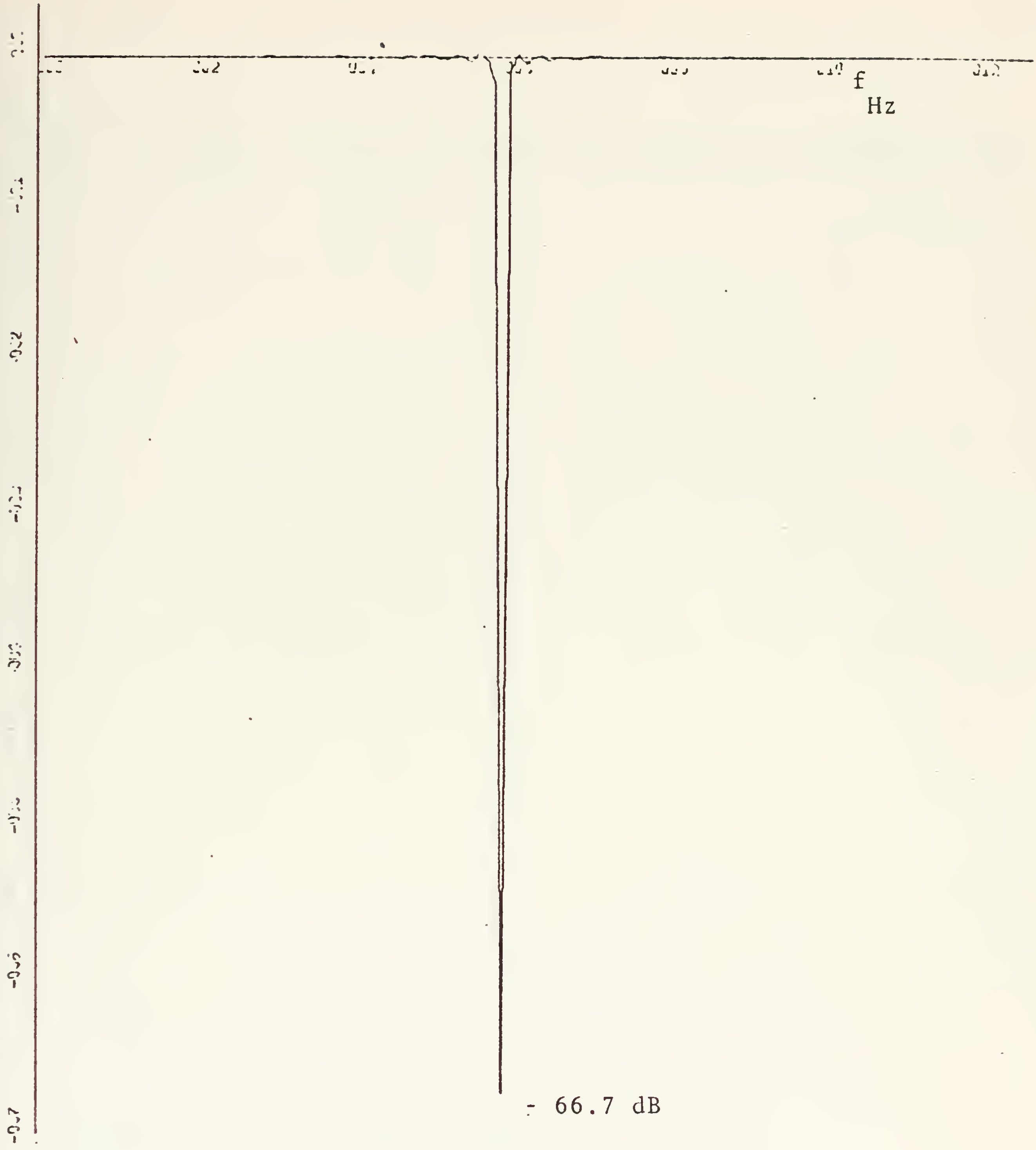
Table (4-2) Third order notch-filter.

The results are based on the steady-state frequency response of 10,000 iterations.



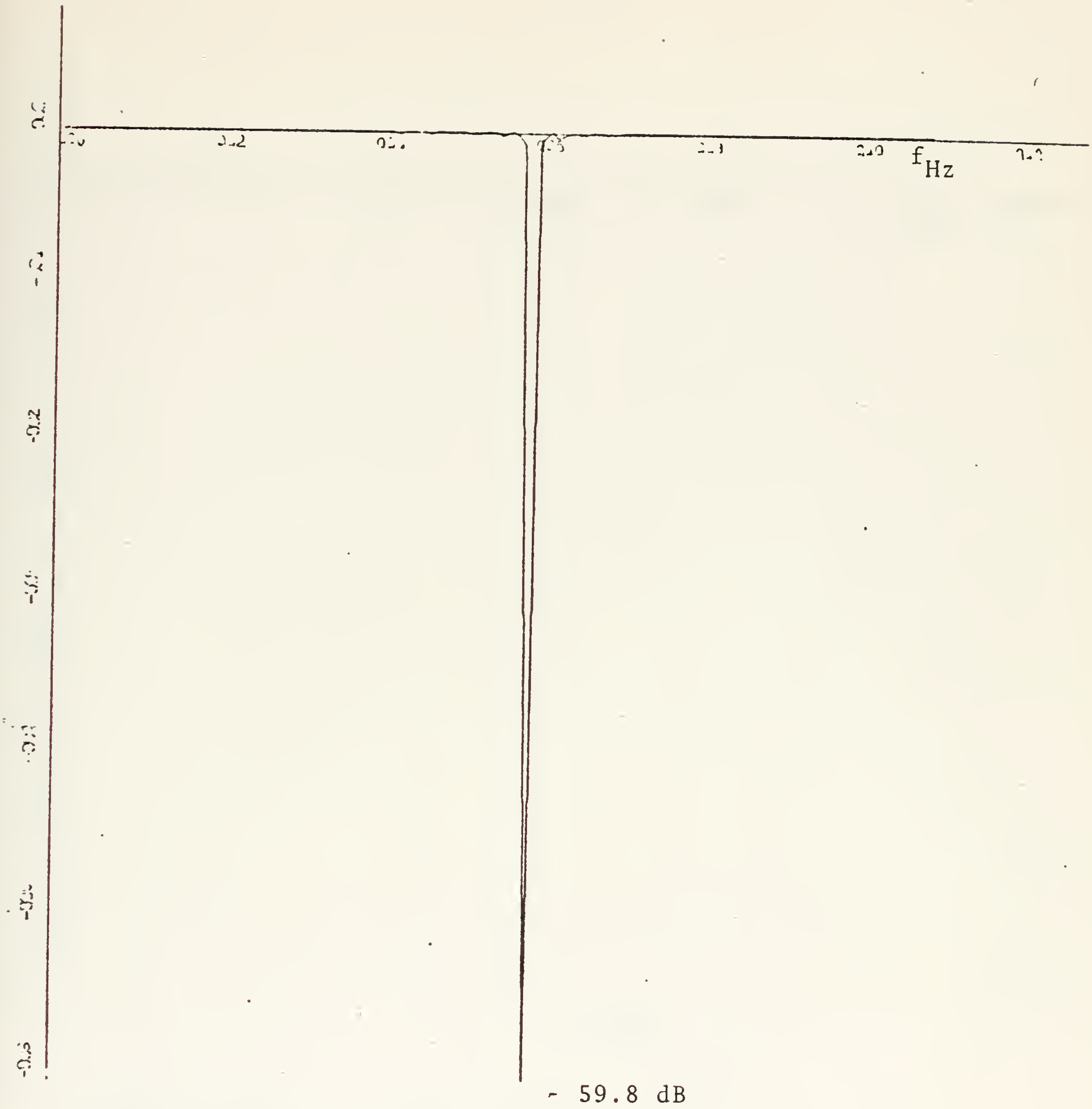
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-32) Steady-state frequency response of third order notch-filter. a=0.95



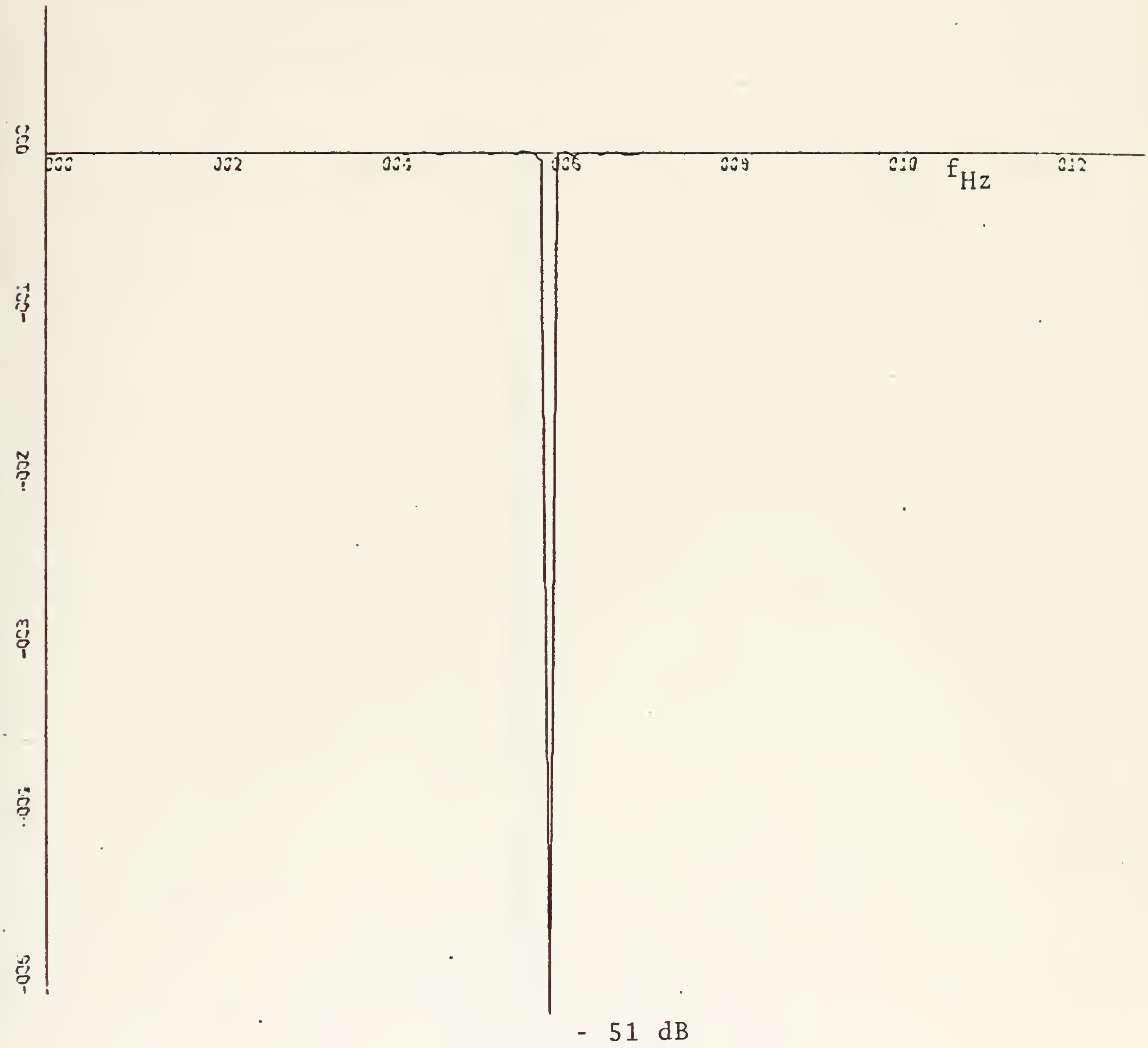
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-33) Steady-state frequency response of third order notch-filter. a=0.985



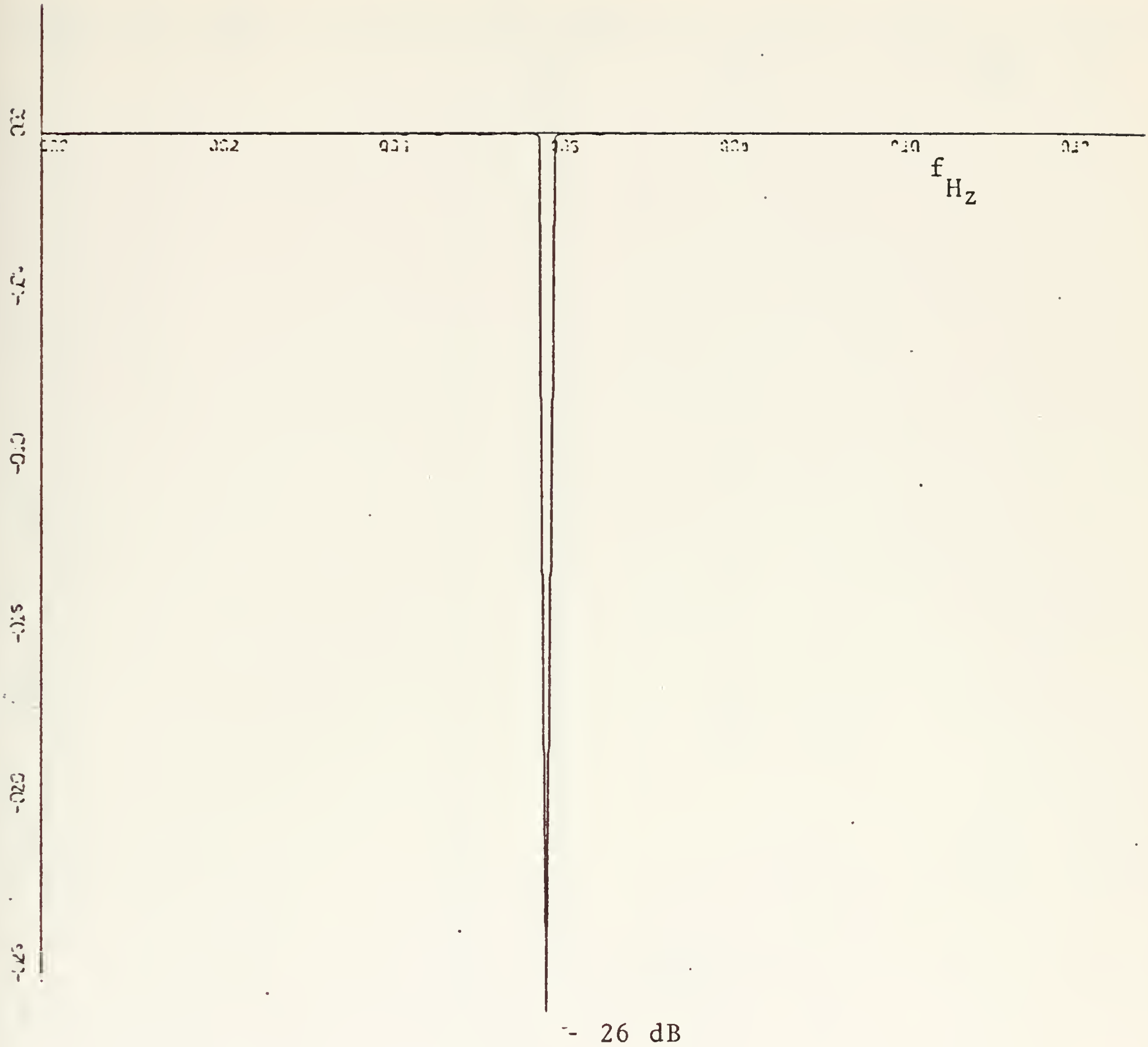
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-34) Steady-state frequency response of third notch-filter. a=0.99



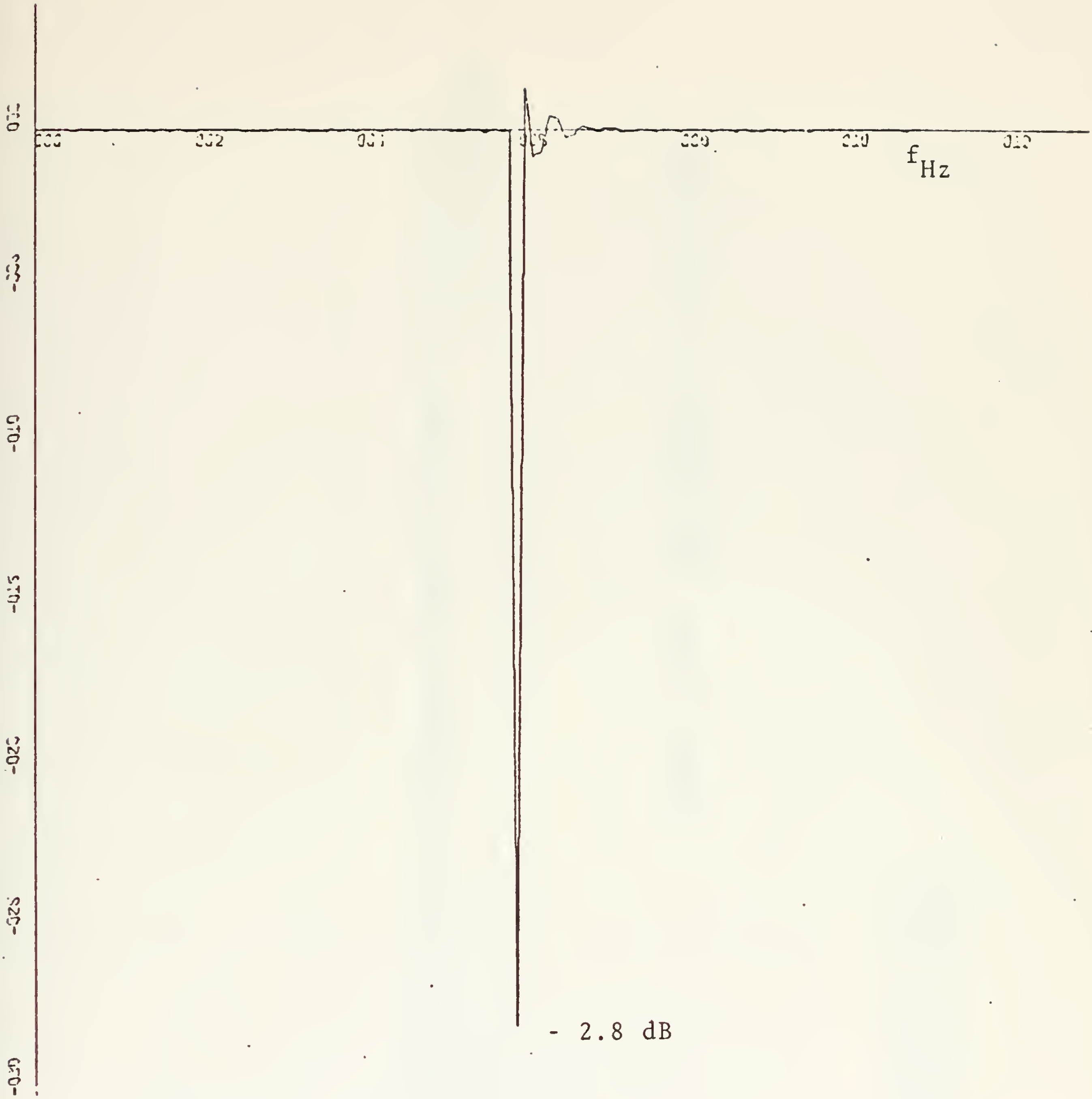
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-35) Steady-state frequency response of third order notch-filter. a=0.995



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=5.00E+00 UNITS INCH.

Figure (4-36) Staedy-state frequency response of third order notch-filter. a=0.999



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

Figure (4-37) Staedy-state frequency response of third order notch-filter. a=0.9999

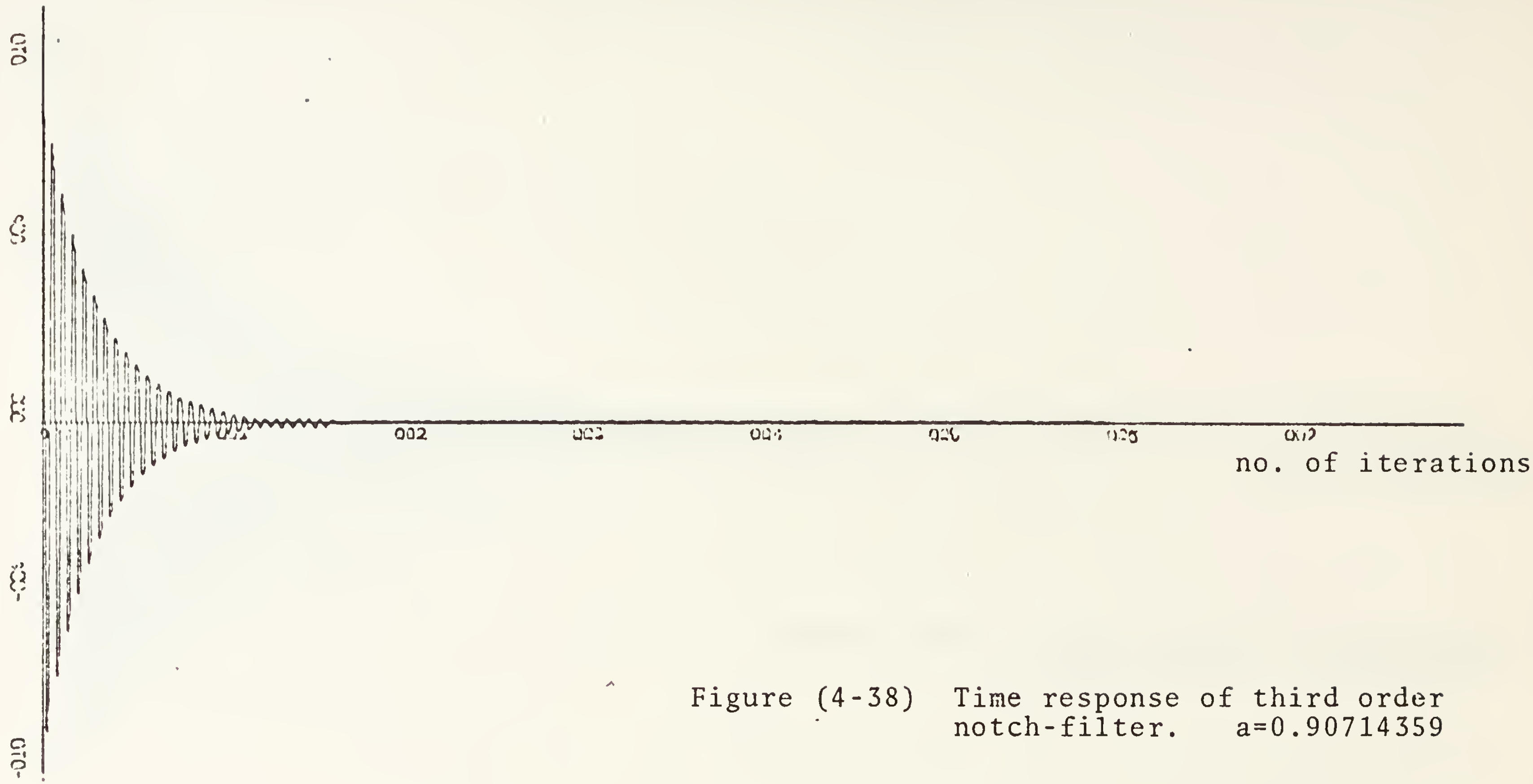


Figure (4-38) Time response of third order notch-filter. a=0.90714359

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

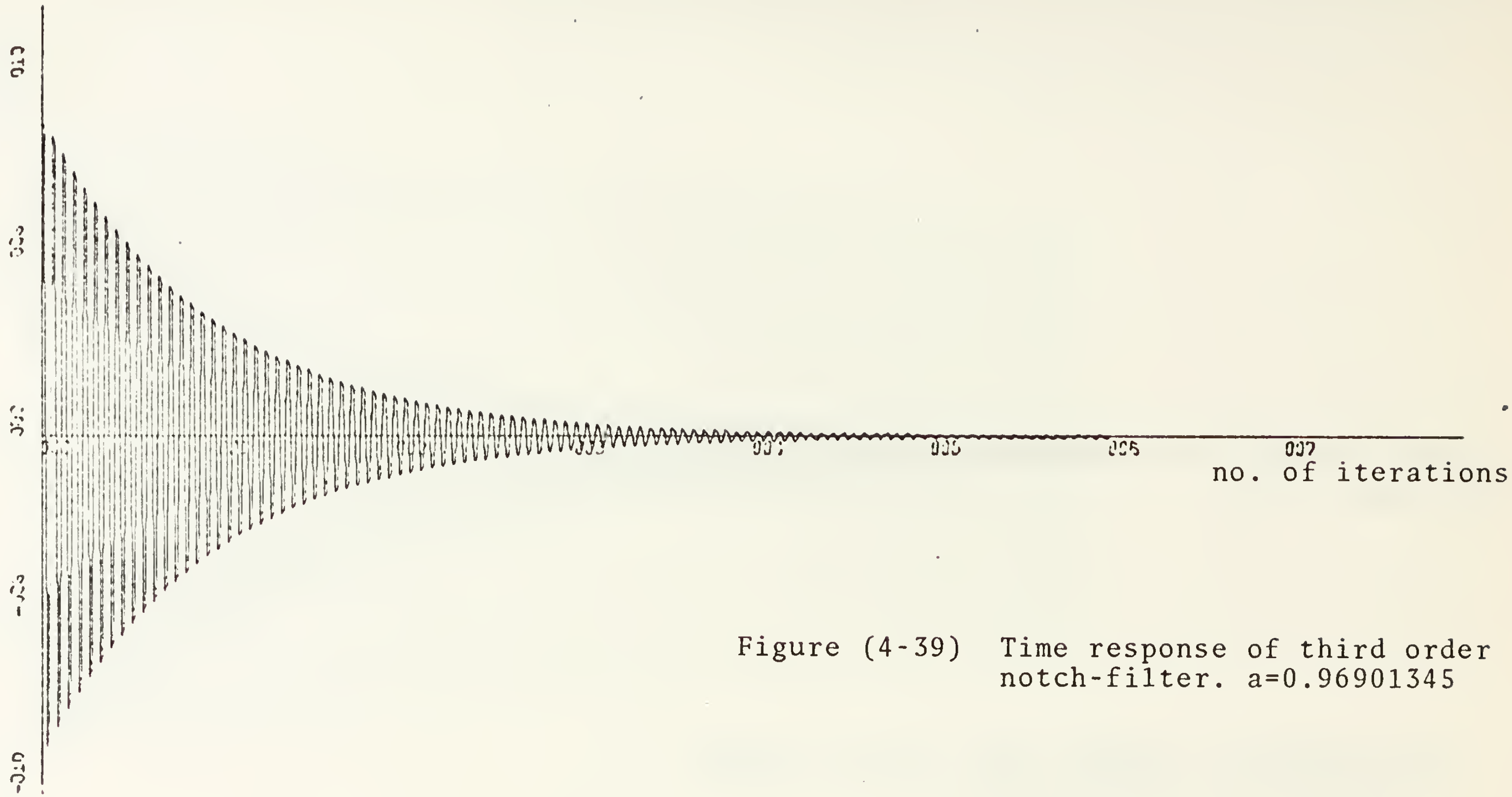


Figure (4-39) Time response of third order notch-filter. a=0.96901345

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

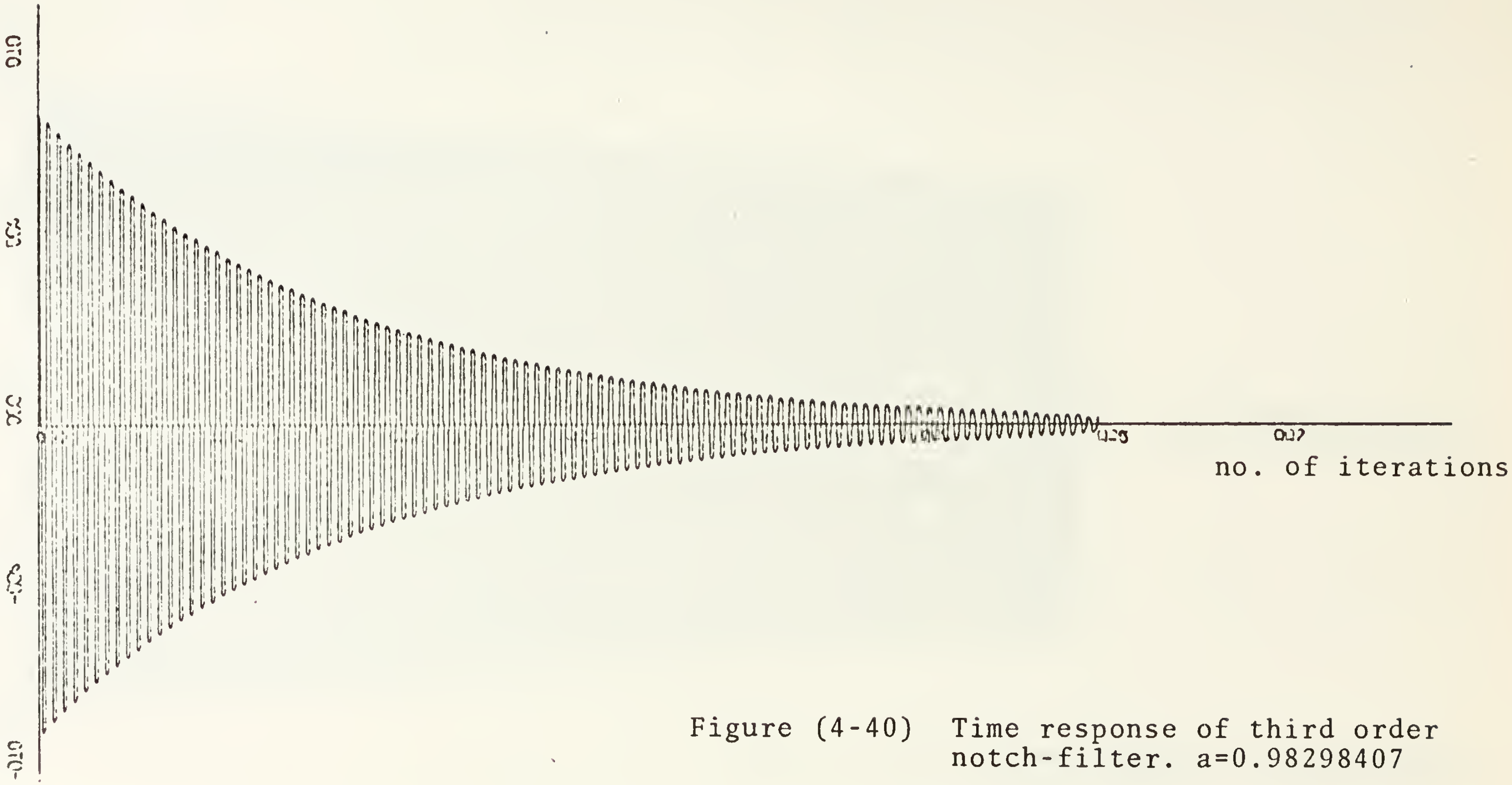


Figure (4-40) Time response of third order notch-filter. a=0.98298407

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

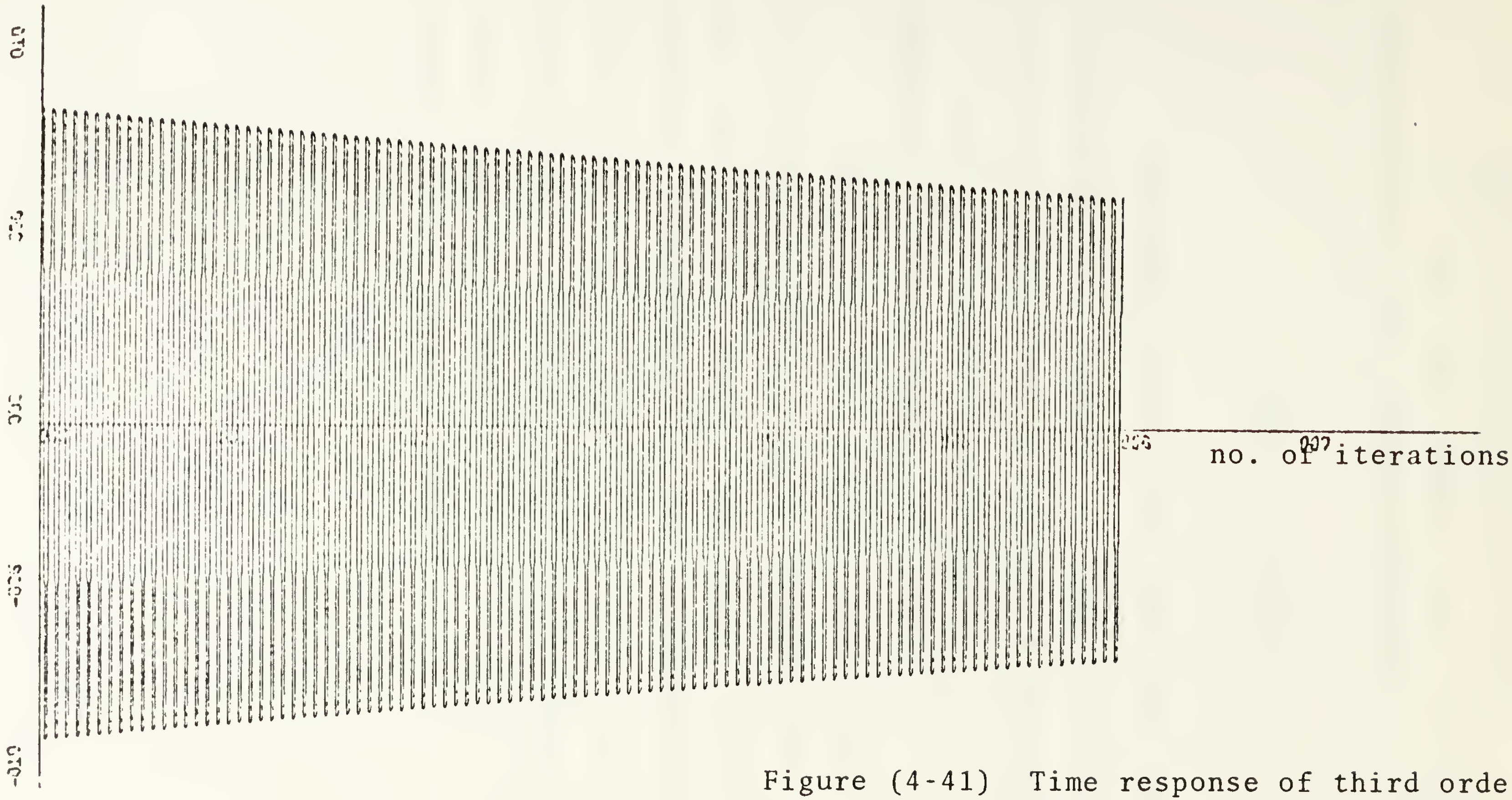


Figure (4-41) Time response of third order notch-filter. a=0.99845153

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

3. Sixth Order Digital Notch-Filter

The sixth order digital notch-filter is characterized by the transfer function

$$H(z) = \frac{1 + z^{-6}}{1 + az^{-6}} \tag{4-6}$$

One often realizes the sixth order digital filter by cascading three sections of second order filters. This is done to reduce the sensitivity of pole positions due to truncation because of round-off error of iteration computations in the simulation program.

The sixth order digital notch-filter characterized by transfer function (4-6) is discussed here. There is only one source of error; the noise produced in the computer implementation is small compared with the general form of sixth order digital notch-filter of thirteen coefficients. Therefore, one can implement the sixth order digital notch-filter (4-6) by direct canonic realization as given by Fig. (4-42).

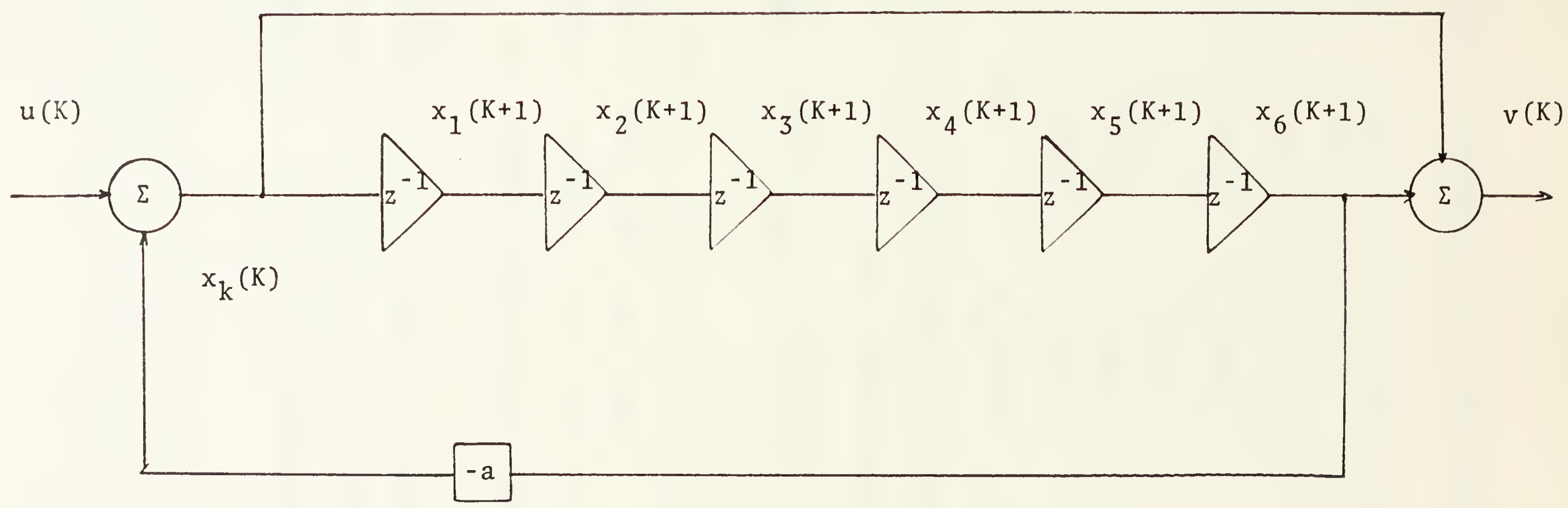


Figure (4-42) Direct canonic realization of sixth order digital notch-filter.

The corresponding state equations are given by:

$$
\begin{aligned}
x_1\ (K) &= -\ a\ x_6\ (K+1) + u\ (K) \\
x_1\ (K+1) &= x_1\ (K) \\
x_2\ (K+1) &= x_1\ (K+1) \\
x_3\ (K+1) &= x_2\ (K+1) \\
x_4\ (K+1) &= x_3\ (K+1) \\
x_5\ (K+1) &= x_4\ (K+1) \\
x_6\ (K+1) &= x_5\ (K+1) \\
v(K) &= x_6\ (K+1) + x_1\ (K)
\end{aligned}
\tag{4-7}
$$

One still requires the notch-frequency at $f_0 = 60$ Hz; the equation (4-1) gives the sampling frequency T as follows

$$T = \frac{1}{2mf_0} = \frac{1}{2 \times 6 \times 60} = 1.388888889 \times 10^{-3} \text{ sec.}$$

It is noted that the sampling time for sixth order is one half of that for the third order case.

Figures (4-43) to (4-46) illustrate the time response of a sixth order digital notch-filter with input sinewave of 60 Hz frequency.

Figures (4-47) to (4-49) are frequency response plots of one thousand, two thousand and two thousand and five hundred iterations, respectively, corresponding coefficient $a = 0.98298407$.

It is noted that the sixth order digital notch-filter requires about fifteen thousand iterations to reach the steady-state. The steady-state gain frequency responses are

illustrated in Figs. (4-50) to (4-55) and some numerical values are tabulated in Table (4-4).

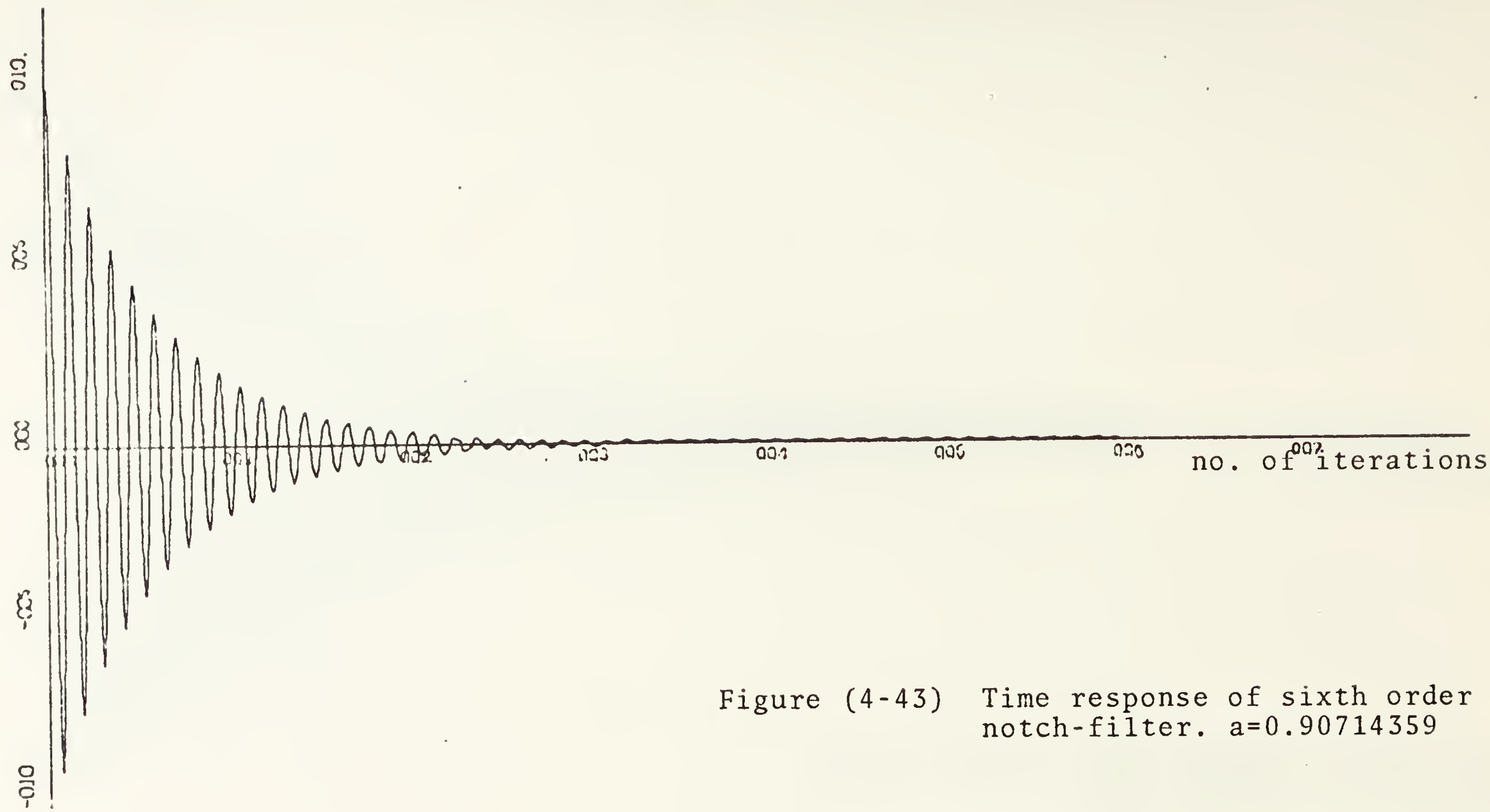


Figure (4-43) Time response of sixth order notch-filter. a=0.90714359

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

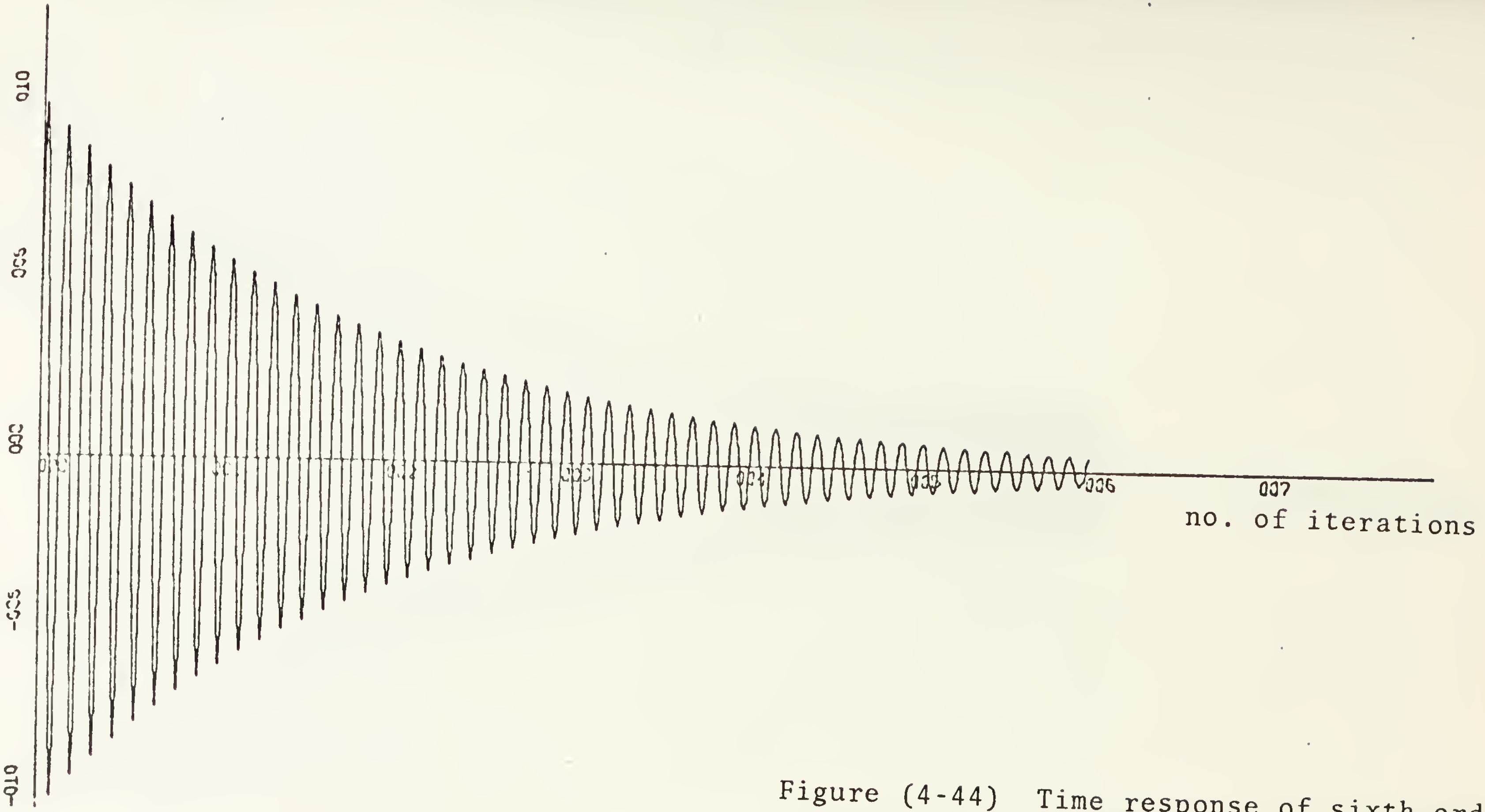


Figure (4-44) Time response of sixth order notch-filter. a=0.96901345

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

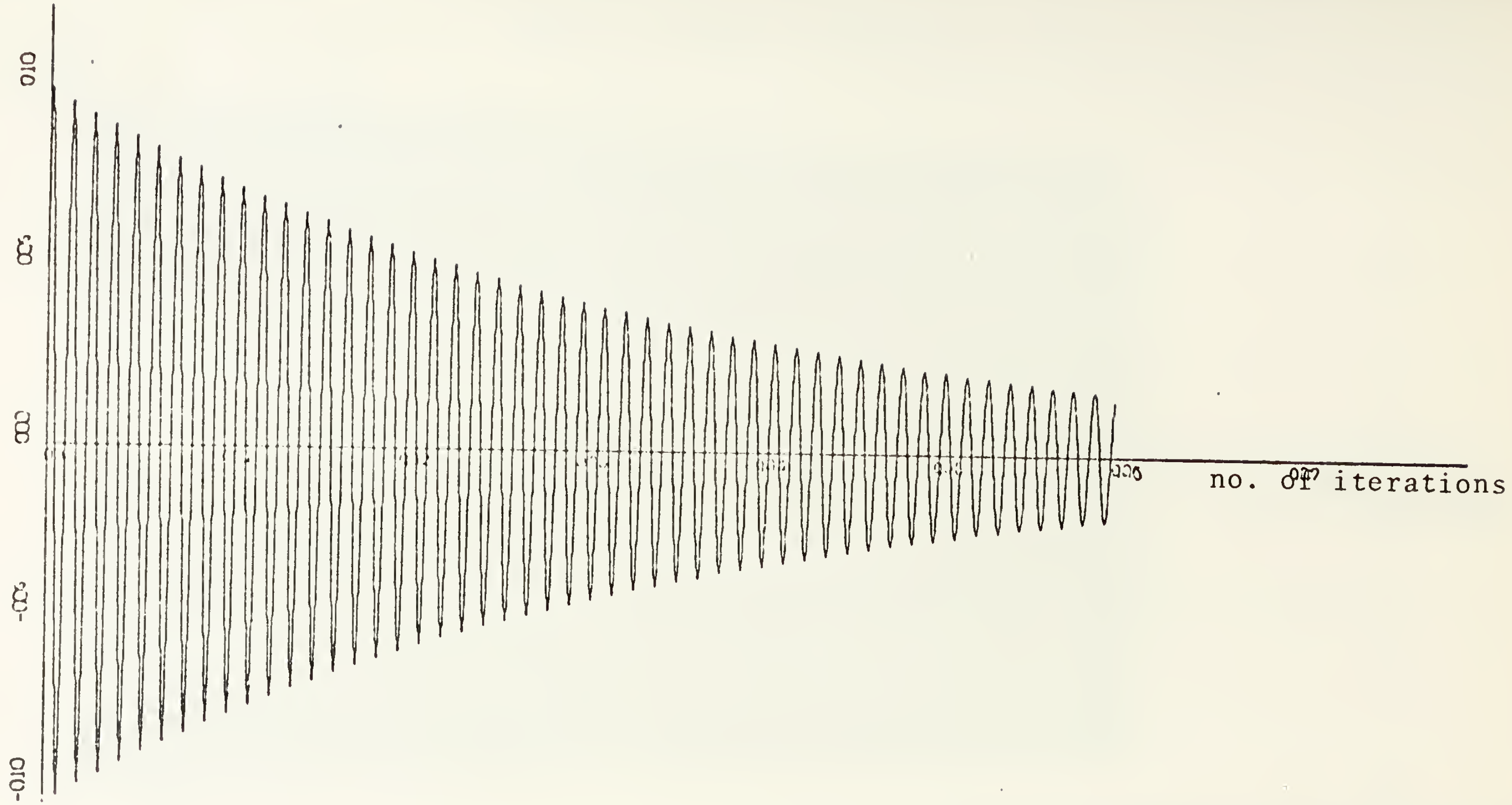


Figure (4-45) Time response of sixth order notch-filter. a=0.98298407

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

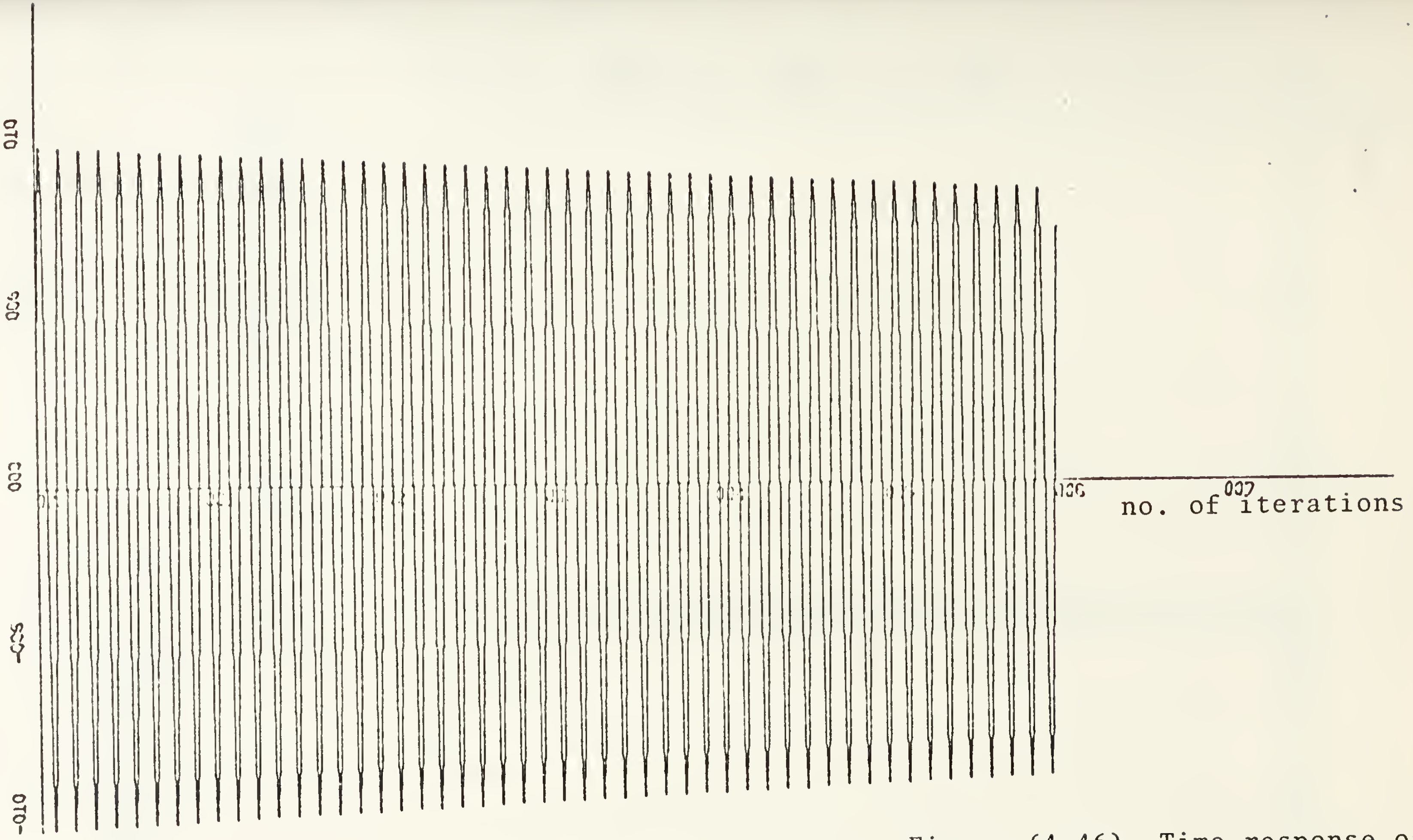
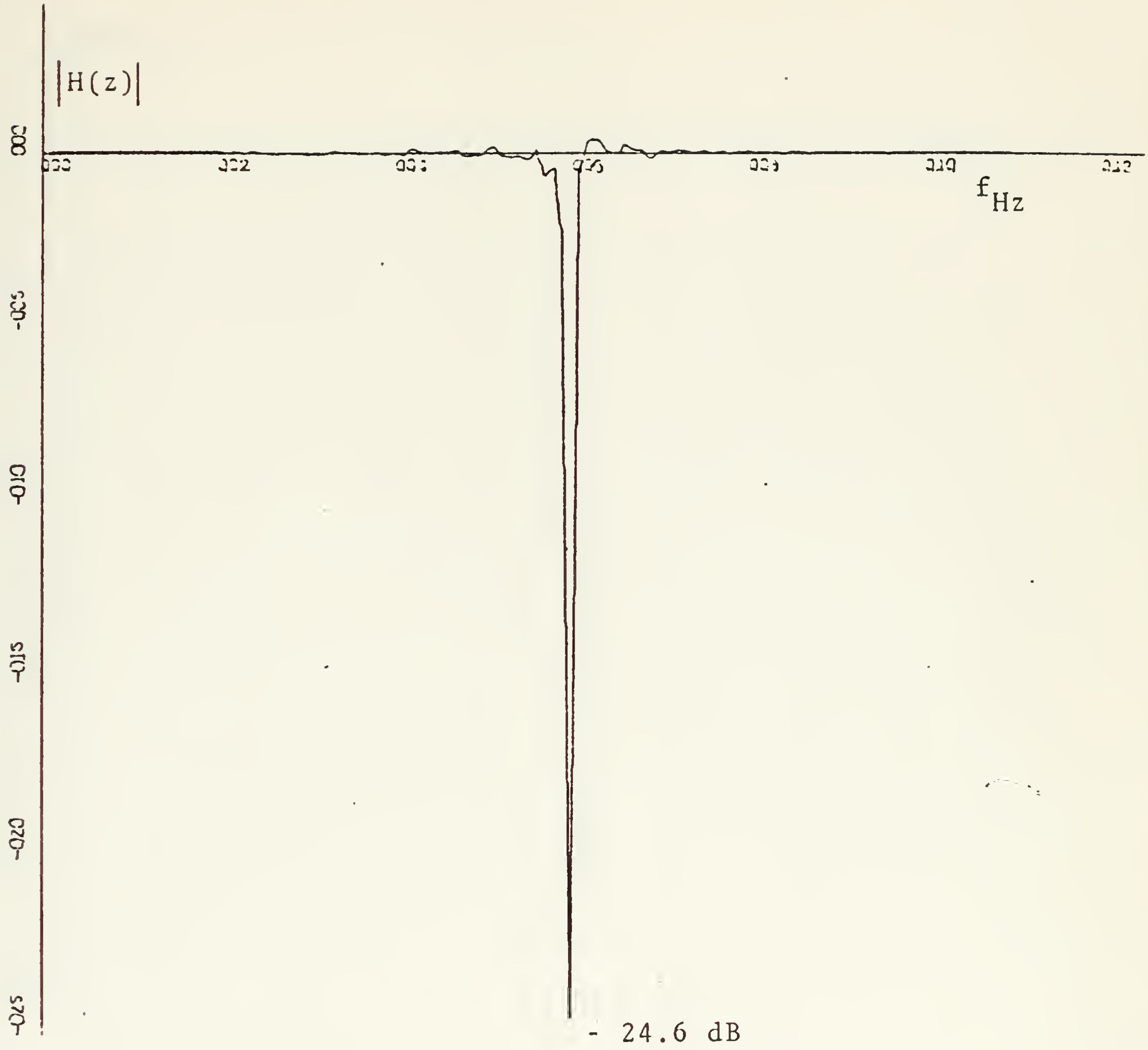


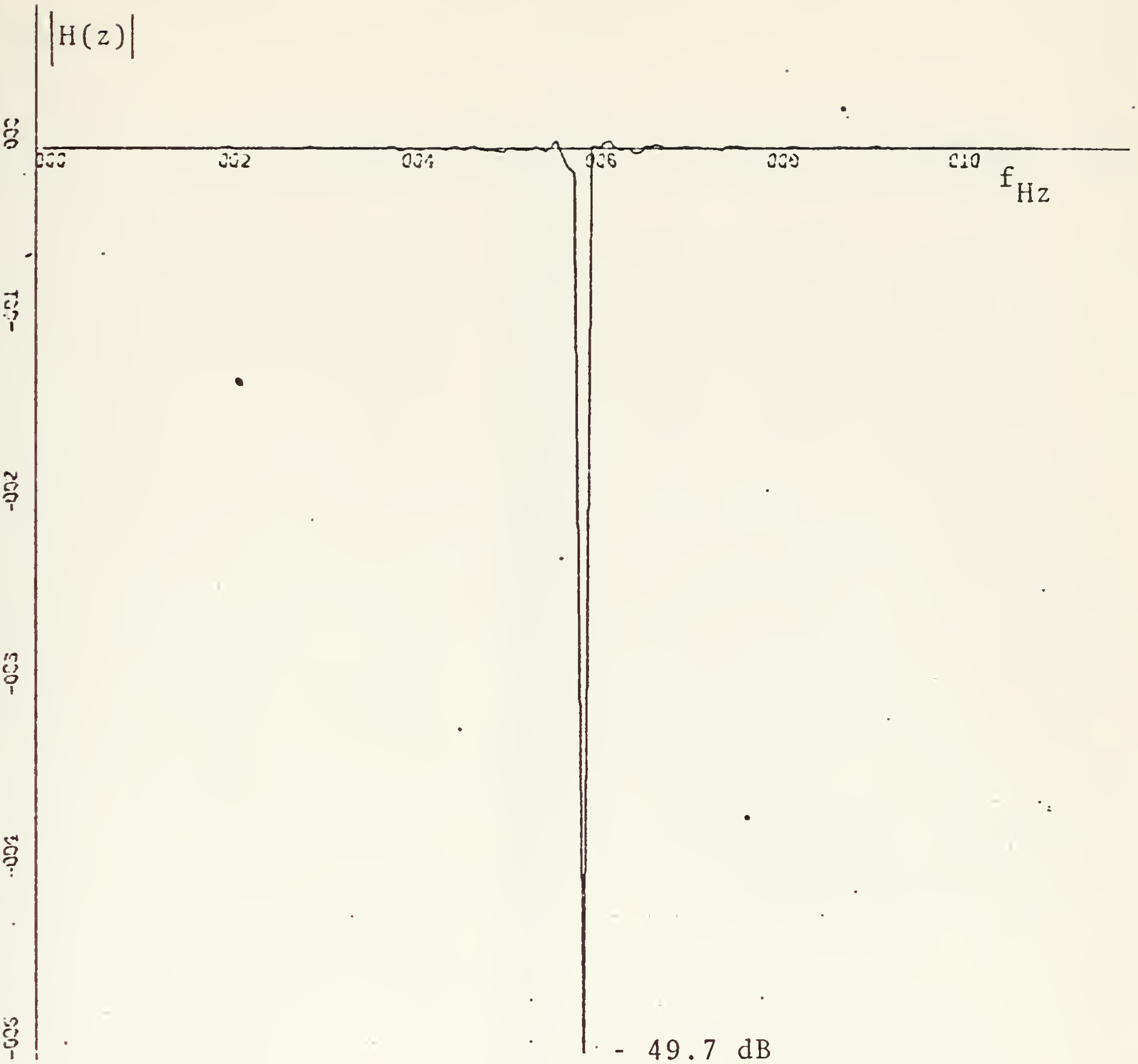
Figure (4-46) Time response ofsixth notch-filter. a=0.99845153

X-SCALE=1.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=5.00E+00 UNITS INCH.

Figure (4-47) Sixth order digital notch-filter at 60 Hz notch and 1000 iterations. a=0.98298407



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-48) Sixth order digital notch-filter at 60 Hz notch and 2000 iterations. a=0.98298407

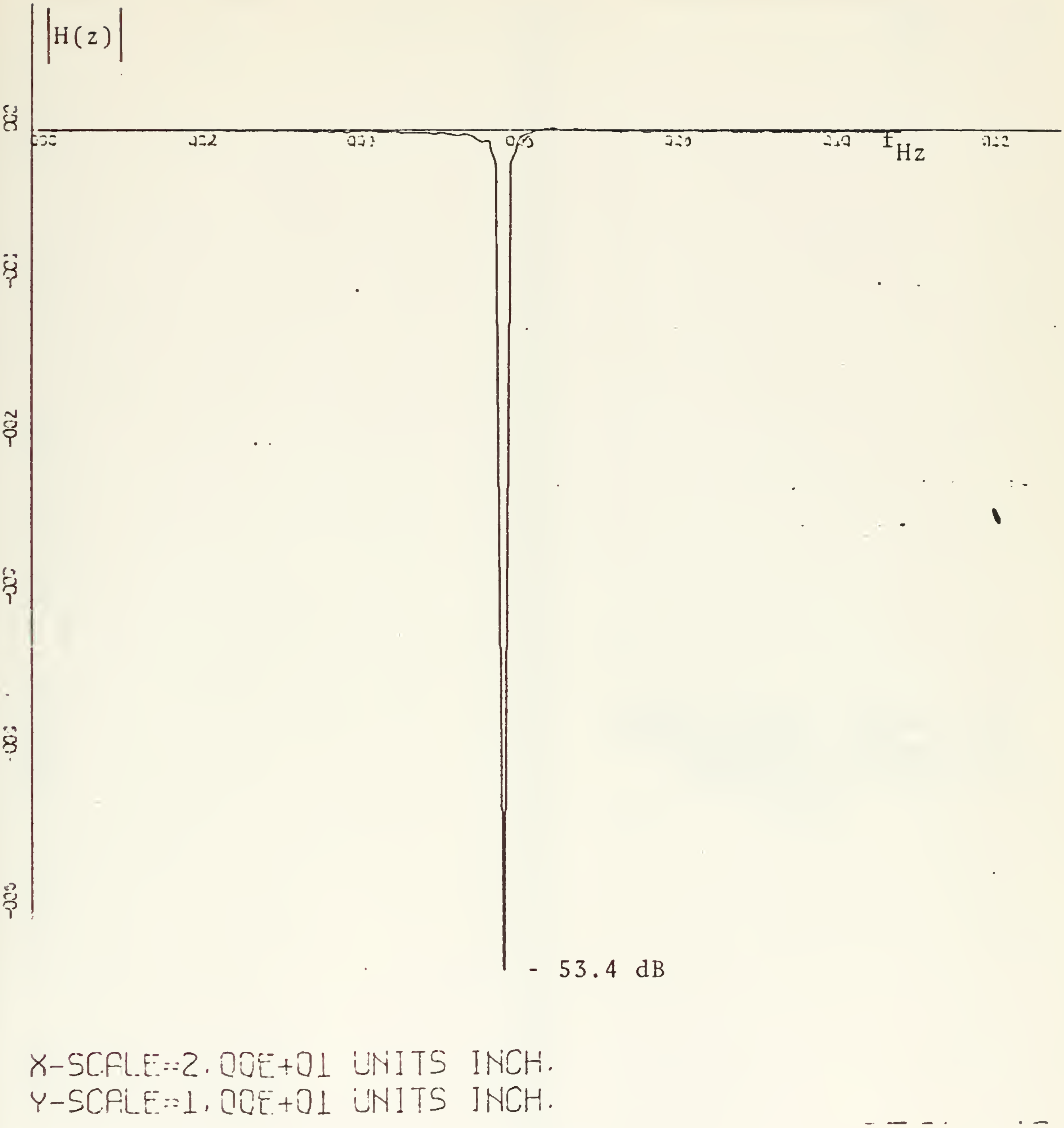


Figure (4-49) Sixth order digital notch-filter at 60 Hz notch and 2500 iterations. a=0.98298407

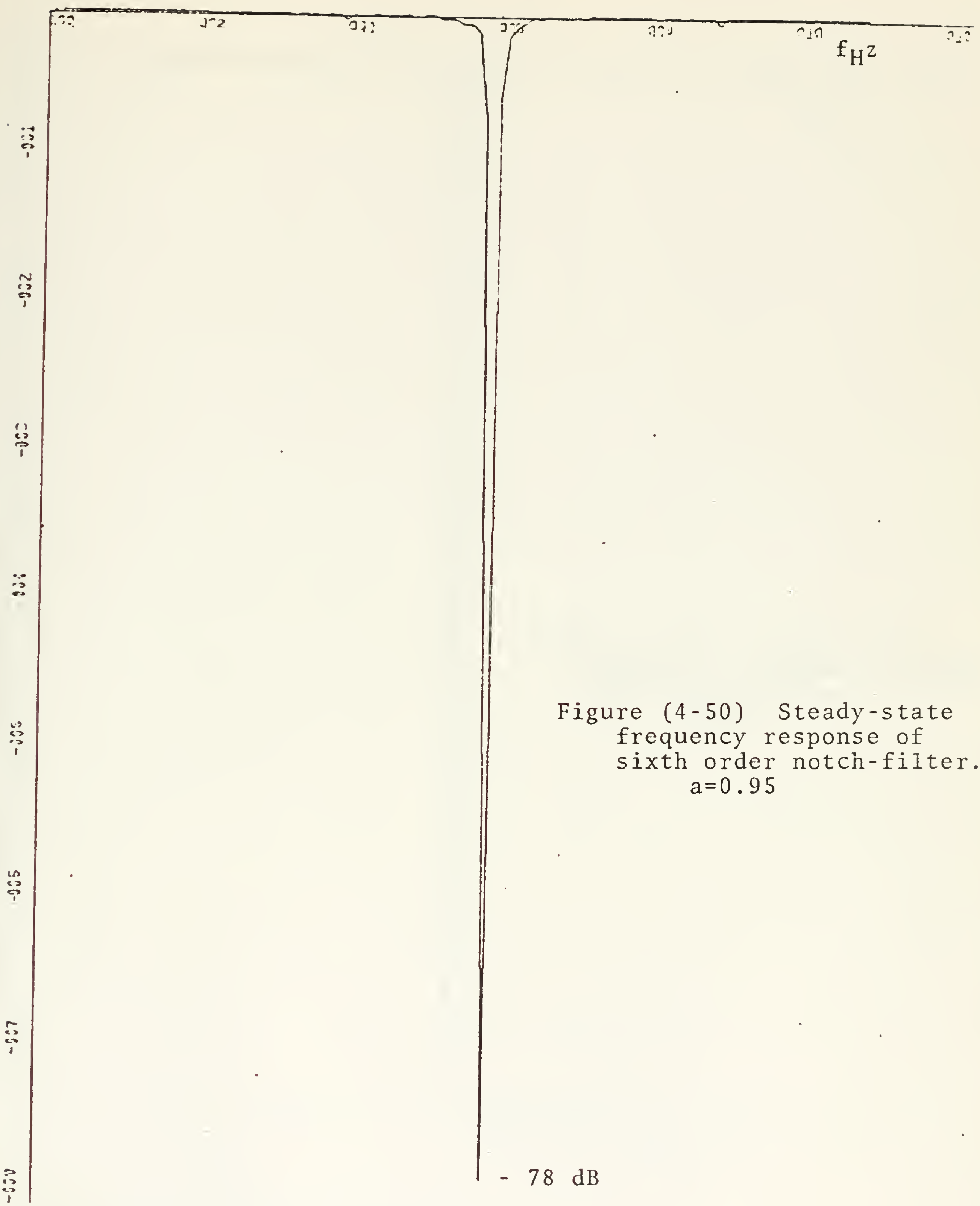


Figure (4-50) Steady-state frequency response of sixth order notch-filter. a=0.95

X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

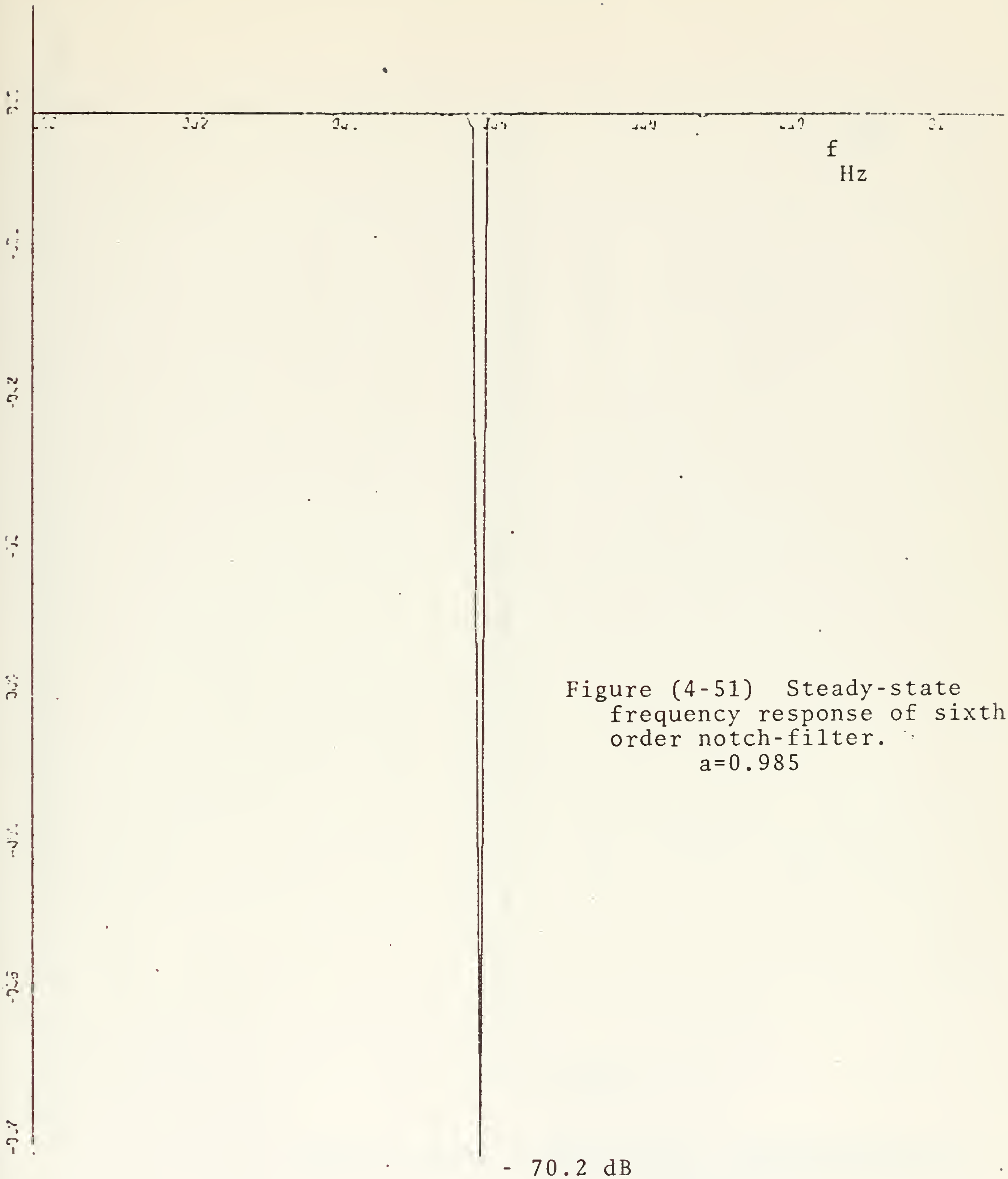


Figure (4-51) Steady-state frequency response of sixth order notch-filter. a=0.985

X-SCALE 2.00E+01 UNITS INCH.
Y-SCALE 1.00E+01 UNITS INCH.

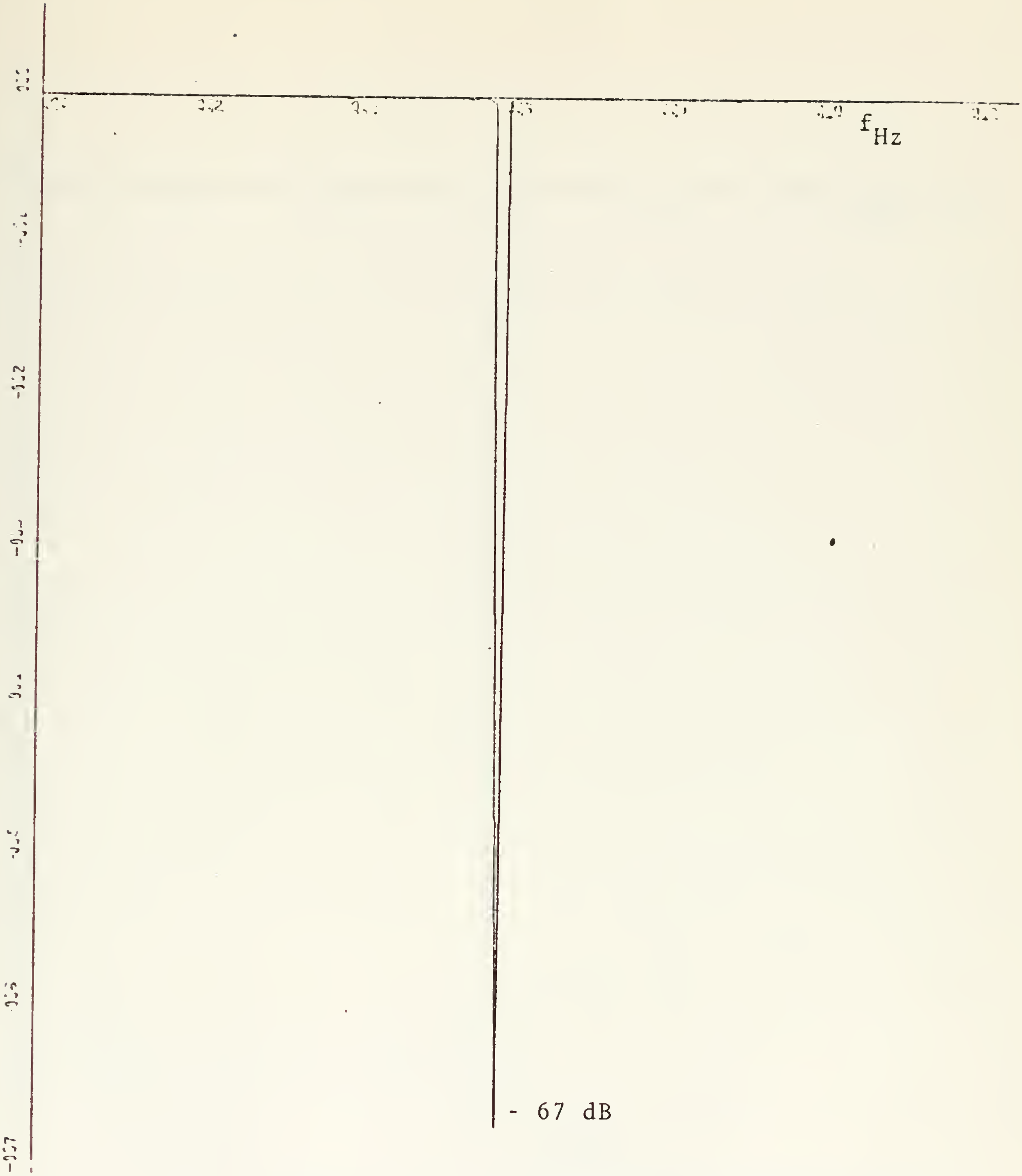
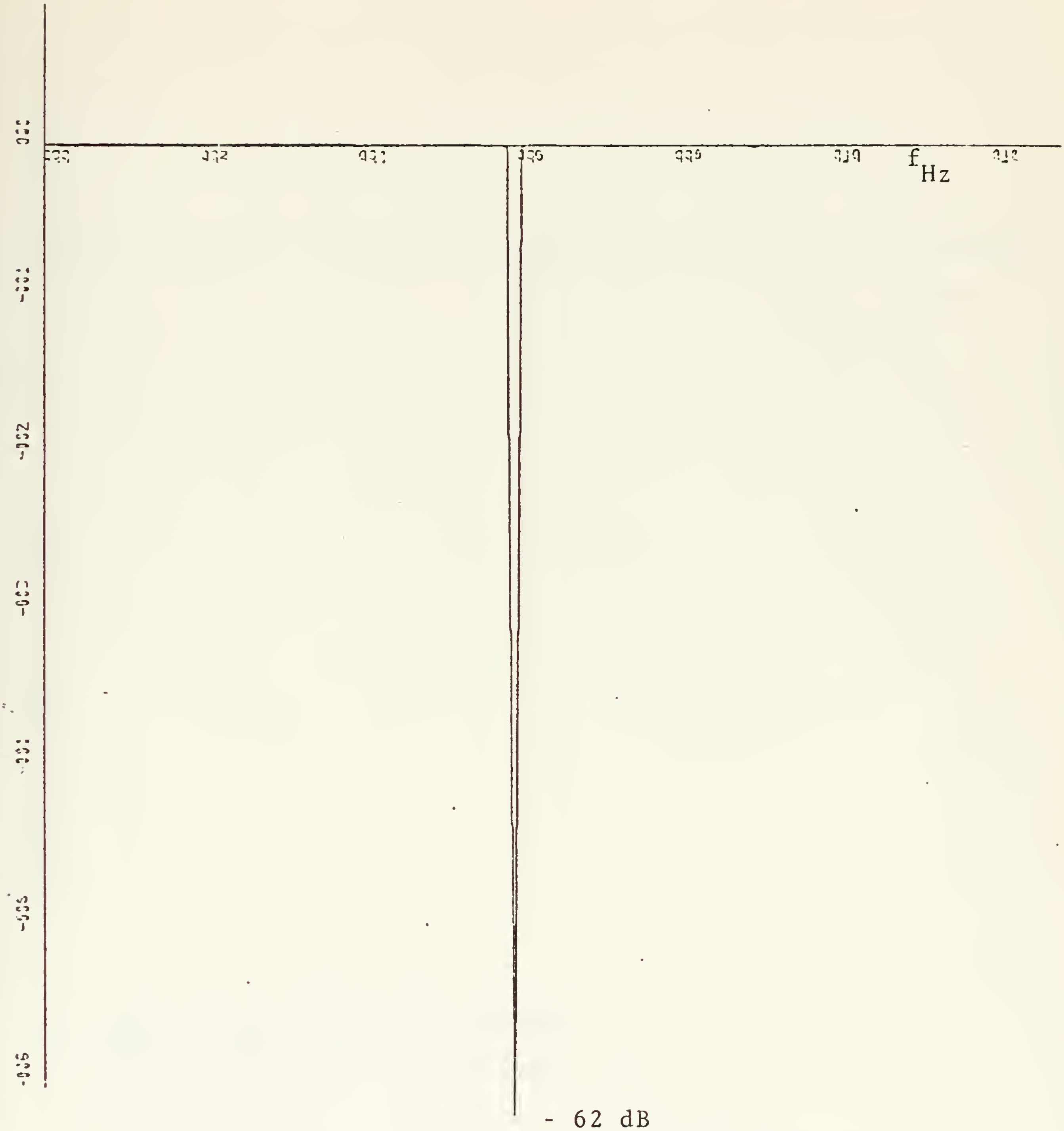


Figure (4-52) Staedy-state frequency response of sixth order notch-filter. a=0.99

X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

Figure (4-53) Steady-state frequency response of sixth order notch-filter.
a=0.995

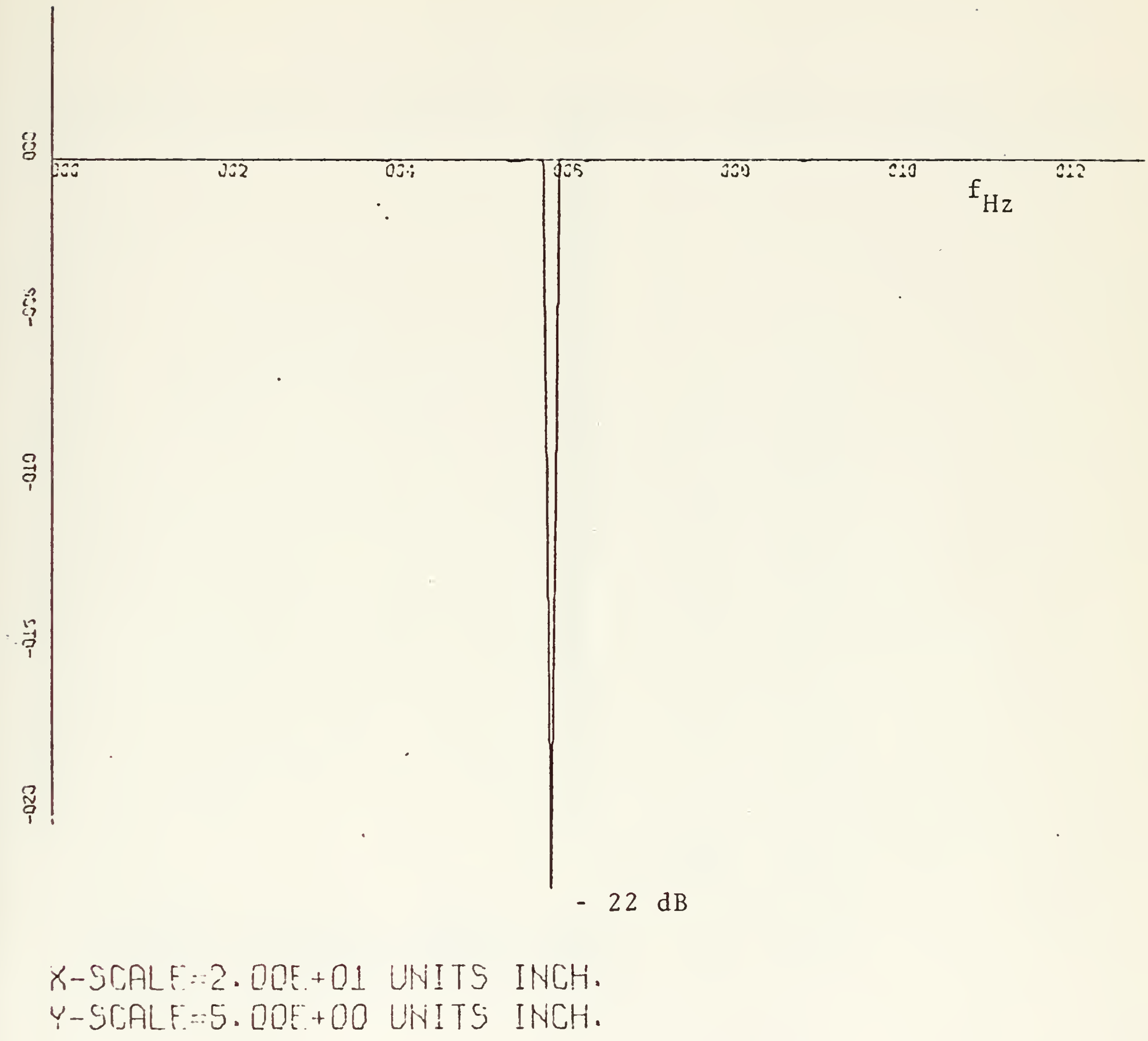
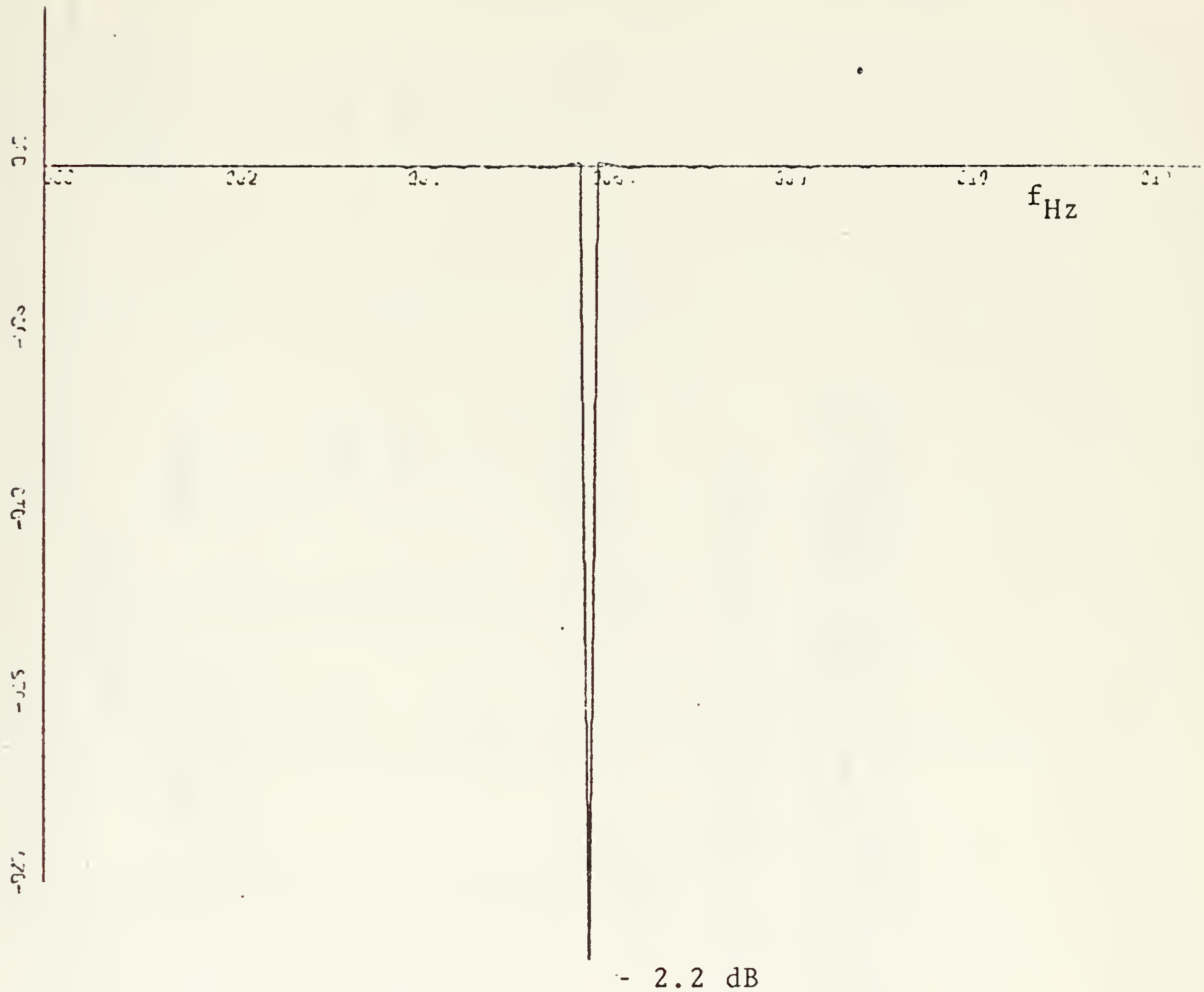


Figure (4-54) Staedy-state frequency response of sixth order notch-filter.
a=0.999



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=5.00E 01 UNITS INCH.

Figure (4-55) Steady-state frequency response of sixth order notch-filter. a=0.9999

COEFFICIENT	NOTCH GAIN in dB	NOTCH-WIDTH in Hz		MAXIMUM PASSBAND RIPPLE in dB
		PRACTICE	THEORY	
0.95	- 78	2	0.2	1.24
0.985	- 70	< 1	0.06	0.1
0.99	- 67	< 1	0.04	0.06
0.995	- 62	< 1	0.02	0.05
0.999	- 22	< 1	0.007	0.02
0.9999	- 2.2	< 1	0.006	0.007

Table (4-3) Sixth order notch-filter.

The results are based on the steady-state frequency response of 15,000 iterations.

One is lead to attempt to realize the sixth order digital notch-filter by cascading three sections of second order as follows:

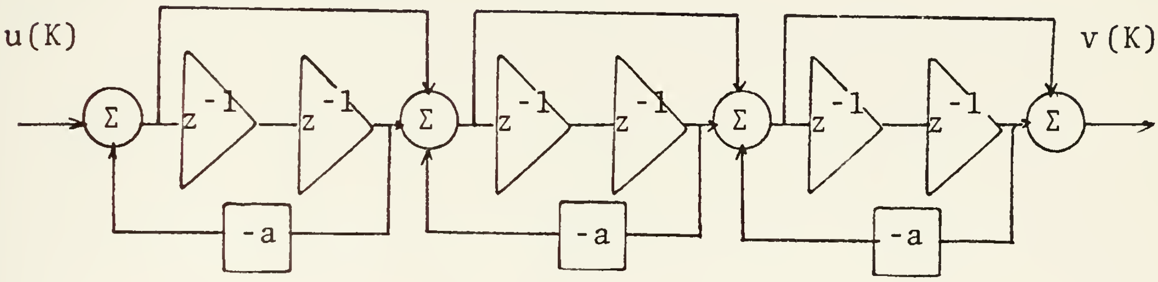


Figure (4-56) Realization of sixth order digital notch-filter by cascade.

The steady-state gain frequency response is plotted in Fig. (4-57) and (4-58) for coefficient a=0.99 and a=0.995, respectively.

Compare with the results of direct realization of sixth order notch-filter in Figs. (4-52) and (4-53); the notch-gain in realization by cascading is deeper than in direct realization. But in passband it does not improve anything. It seems worse than direct form by introducing more noise; one can see gain frequency response behavior for coefficient a=0.9999999 between them in Figs. (4-59) and (4-60).

It proves that the result is contrary with the statement of many current lectures: the pole-position of direct form is sensible than cascade form in realizing high order digital filter [Ref. 1]. The result in

Figs. (4-59) and (4-60) states that: the pole-position in cascade form is much more sensible than direct form in realizing sixth order digital notch-filter characterized by the transfer function (2-16).

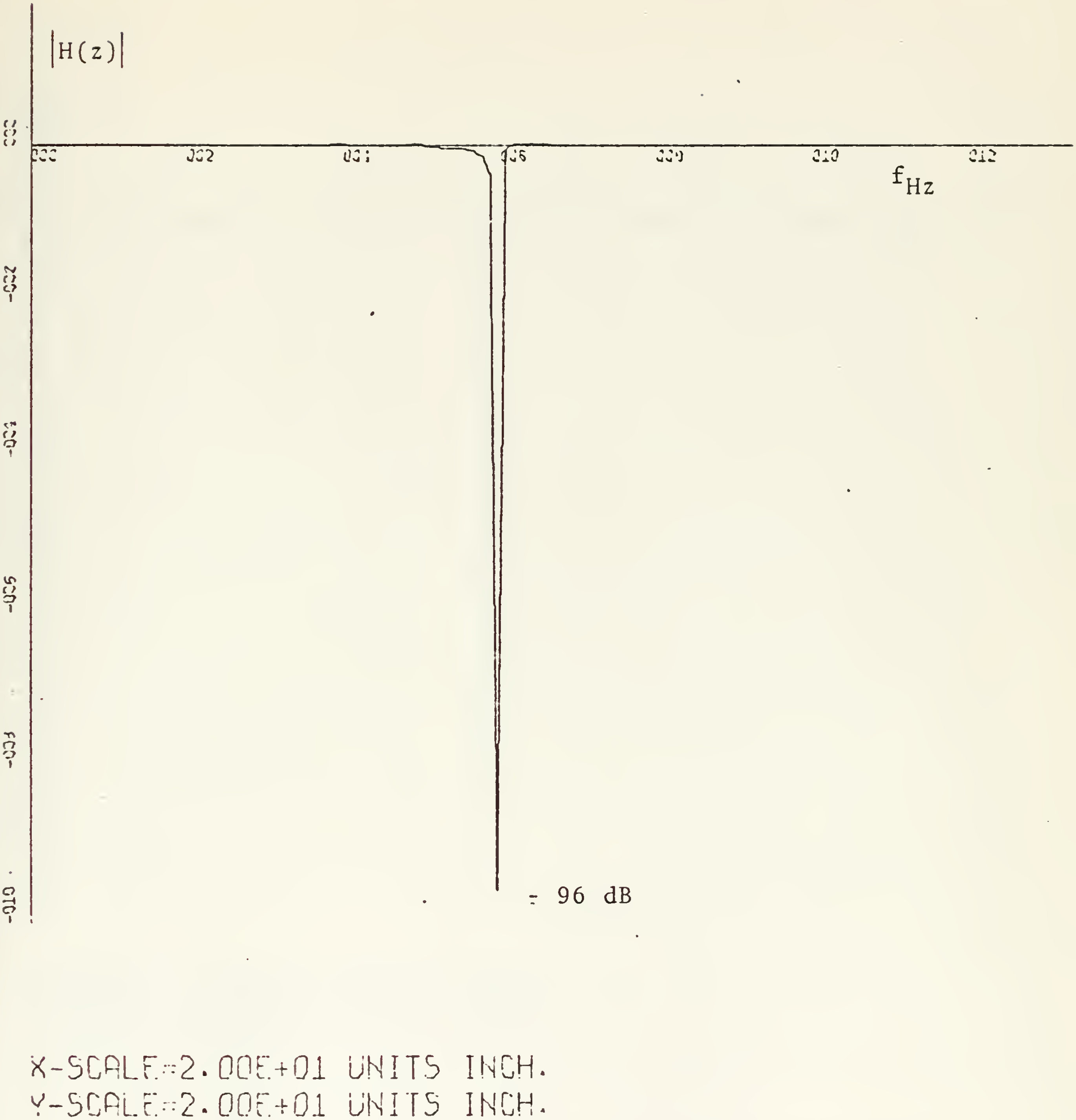
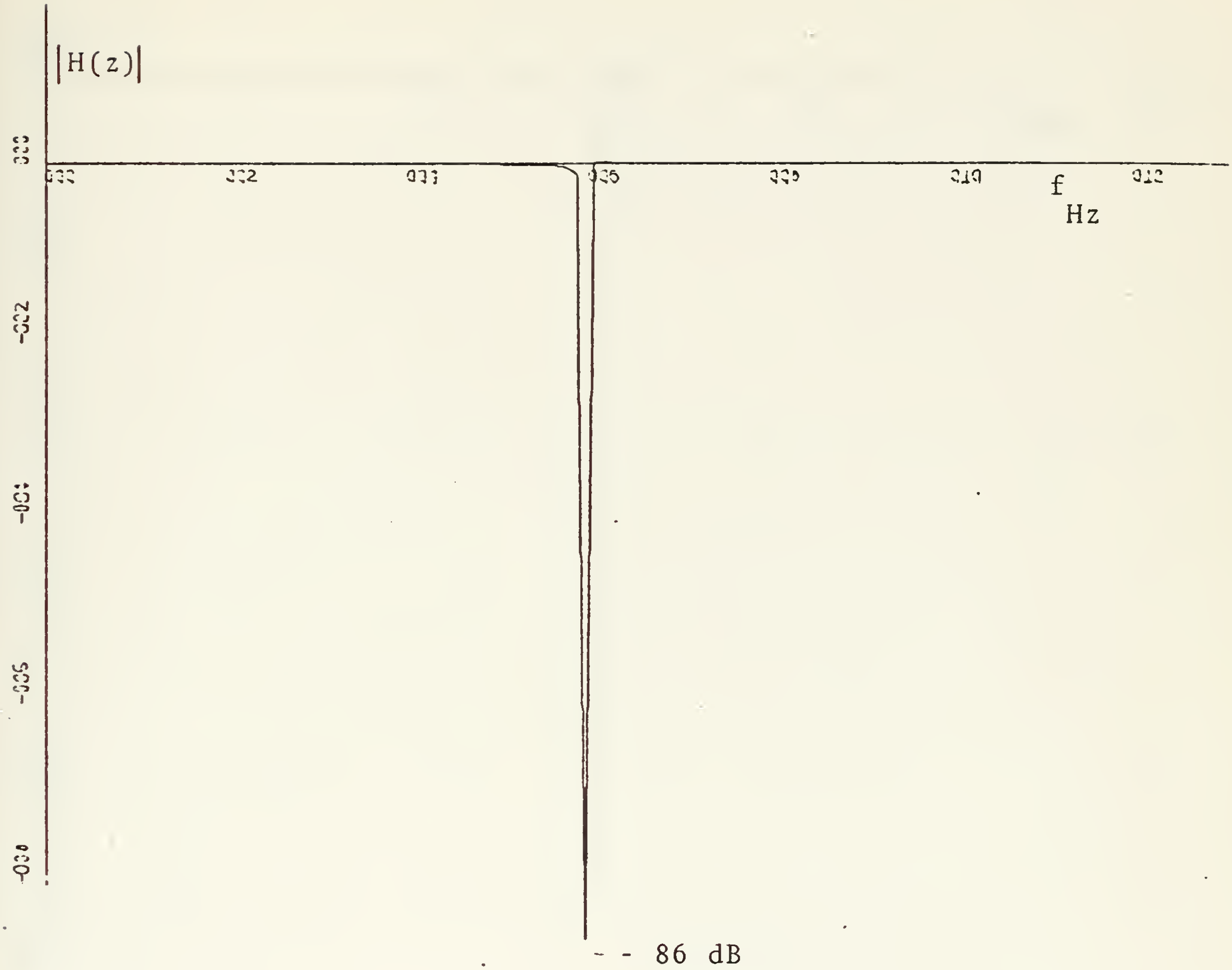
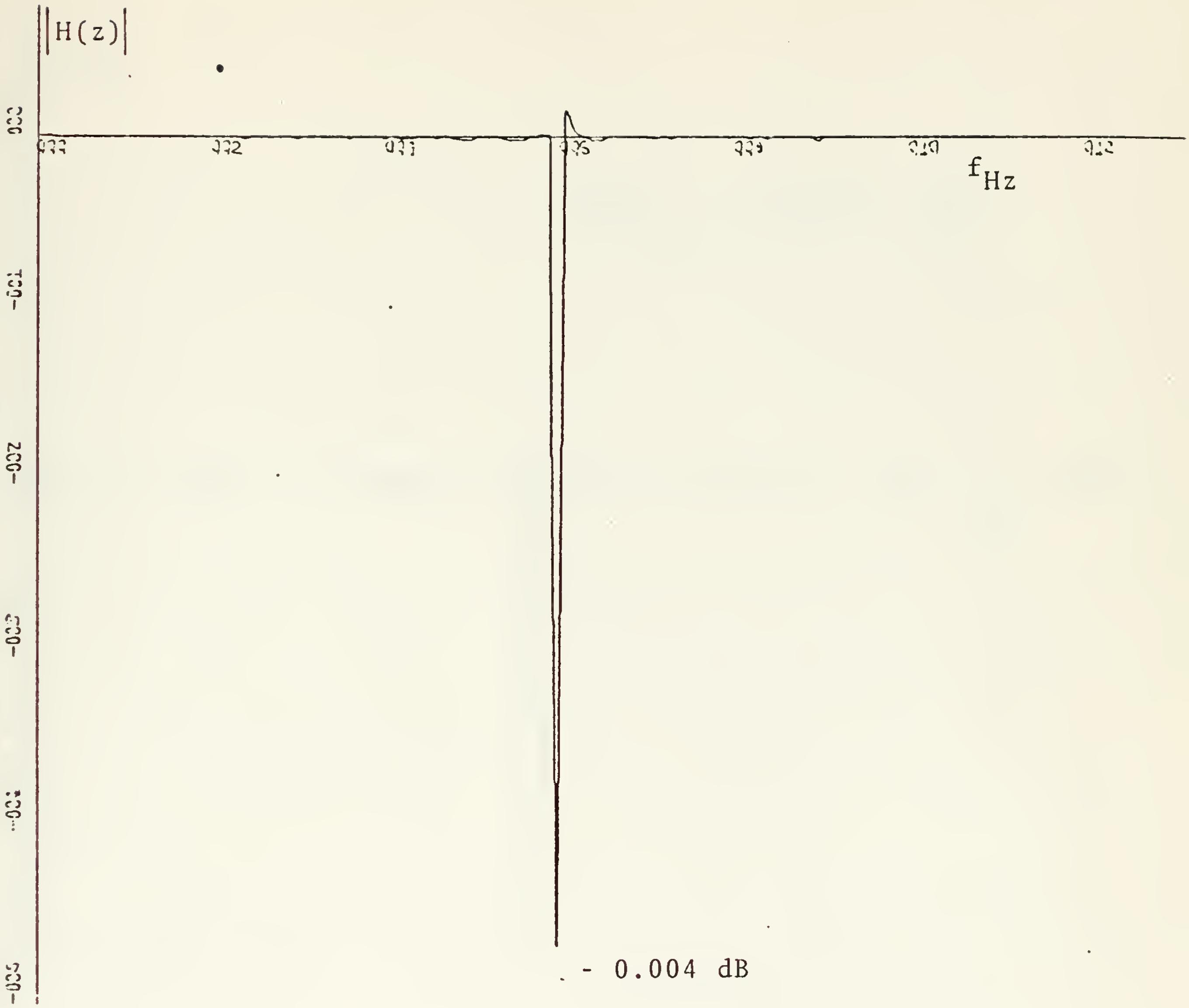


Figure (4-57) Frequency response of sixth order notch-filter by cascade.
a=0.99



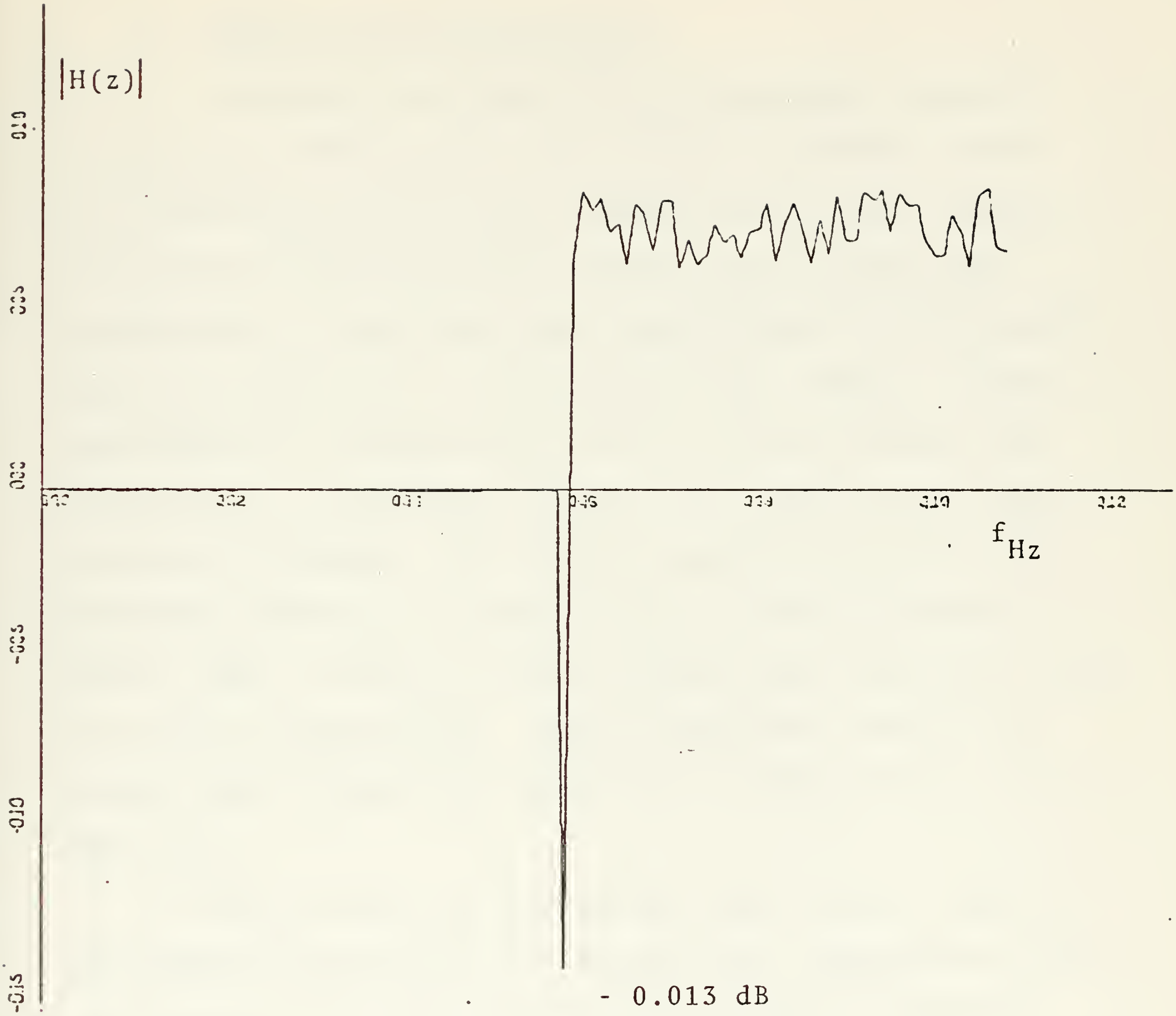
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+01 UNITS INCH.

Figure (4-58) Frequency response of sixth order notch-filter by cascade.
a=0.995



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=1.00E-03 UNITS INCH.

Figure (4-59) Frequency response of sixth order notch-filter by direct form. a=0.9999999



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=5.00E-03 UNITS INCH.

Figure (4-60) Frequency response of sixth order notch-filter by cascade form.
a=0.9999999

4. Observation and Discussion

In theory, by inspection, the magnitude response in equation (2-9) and the notch-width in equation (2-19) vs. coefficient of m-th order digital notch-filter characterized by transfer function (2-16), one sees that when coefficient, a, approaches one, the magnitude of notch-gain and notch-width will be smaller [see Fig. (4-11)]. From the magnitude response plots, it is seen that theory and practice are in agreement whenever the steady-state output response is reached. This can be seen in the steady-state frequency responses in Figs. (4-12) to (4-17) of second order, Figs. (4-32) to (4-37) of third order and Figs. (4-50) to (4-55) of sixth order. Some of numerical results of steady-state response are tabulated in Tables (4-1), (4-2) and (4-3).

For the case of intermediate state (e.g., state of one thousand iterations--the steady-state responses are not yet reached), the theory and practice appear to disagree. One can see this in frequency response plots in Figs. (4-2) to (4-10) of second order. These phenomena are a result of the random errors produced by truncating coefficients and round-off error in iteration computations in the simulating program due to finite register length of the computer. In this case, it is due to limitations of single precision arithmetic on IBM 360. A second explanation of the apparent deviation from the theoretical stems from error in computation of notch-gain in computer program:

$$Gain_{AB} = 20 \log_{10} \left(\left| \frac{\text{Peak of input}}{\text{Peak of output}} \right| \right)$$

The peak of input is always one, but in the computer program, the peak of input and output are picked simultaneously and it is difficult to choose the sampling time such that it exactly corresponds to input peak; this is demonstrated in Fig. (4-56)

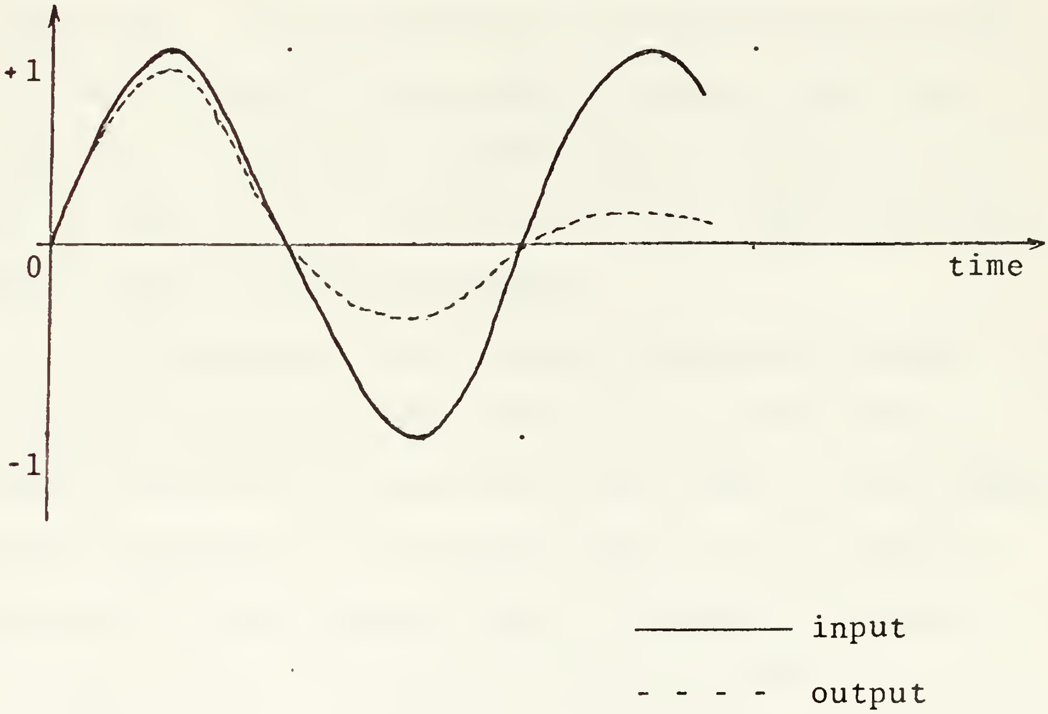


Figure (4-56).

where the sampling time T is fixed for prescribed magnitude response of desired notch-filter.

Suppose:

the peak of input $\simeq$ 0.9

the peak of output $\simeq$ 0.0002

The corresponding notch-gain will be:

GAIN = - 73.06 dB.

Now let the peak close to one; suppose:

the peak of input $\simeq$ 0.99

the peak of output $\simeq$ 0.0001

GAIN = - 79.91 dB.

The error thus realized approximately 6 dB for the notch-gain. By the difficulty of picking exactly the peak of input as well as the peak of output, the actual notch-gains are certainly somewhat deeper than the values listed in Tables (4-1), (4-2), and (4-3). Also, the actual passband ripples should be smoother.

It is known that the second order needs around six thousand iterations, the third order needs around ten thousand iterations and the sixth order needs around fifteen thousand iterations to the steady-state. It is seen by considering the time-respond plots of second, third and sixth order. It is obviously true since higher order in direct realization requires larger number of delays.

It is noted that the passband-ripple is large around notch-frequency and smoother when going further.

The results show that the higher order of digital notch-filter improves the passband-ripple (i.e., reduced ripple) and notch-gain [i.e., deeper notch. See Tables (4-2), (4-2), and (4-3)] and steady-state frequency response

plots in Figs. (4-50) to (4-55) of second, third and sixth order, respectively.

The sixth order notch-filter requires about four minutes to plot one steady-state gain frequency response using fifteen thousand iterations.

One will see that the results in this chapter will give the easy way to work in optimum problems due to finite register length on computer in the next chapter.

B. CCD IMPLEMENTATION*

The charge-coupled device (CCD) was invented at the Bell Telephone Laboratory by Drs. Boyle and Smith in 1970. This new development has been receiving much attention by industry because of its important properties--low power consumption, low noise, and delay capability. Many successful applications have been investigated such as digital signal processing especially in design of recursive digital filter. More details of CCD can be seen in many current lectures. The CCD hardware implementation of recursive digital notch-filter is beyond this research. The following is the sample of block diagram of second order CCD digital notch-filter:

*"Charge-coupled Device for Analog Signal Processing--Recursive Filters Study," Engineer's Thesis by Varachai Imsa-ad, Naval Postgraduate School, December 1974.

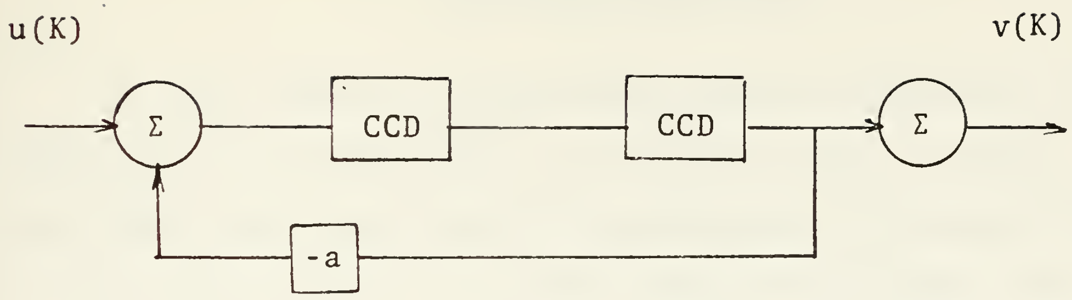


Figure (4-61) Block diagram of a second order CCD digital notch-filter.

V. OPTIMUM COEFFICIENT

In this chapter, a practical method is described for choosing the coefficients of a digital filter to meet arbitrary specifications of the magnitude characteristic.

As shown above, all special purpose computers have finite wordlengths which is a major source of noise in the implementation of digital filters. This is because of the necessity to truncation coefficient length and also because of rounding in the arithmetic operations, especially in the iterative multiplications. This kind of noise determines the signal-to-noise ratio of the system. In some cases, the limitation of wordlength of coefficients may lead to a system which does not satisfy the original conditions.

Especially, in the design of recursive digital notch-filters, one has to select the coefficients such that they satisfy the prescribed characteristic of notch-gain, notch-width, and the smallest passband ripple.

The m-th order digital notch-filter is characterized by the transfer function (2-16). It is seen that the notch-gain, the notch-width, and the passband ripple are functions of a single parameter--the coefficient "a" between 0 and 1. It is possible to construct a performance function of single parameter in the optimization problem.

A. SEARCH PROCEDURE

In the design of an m-th order digital notch-filter, if one wishes the prescribed magnitude having the following characteristics:

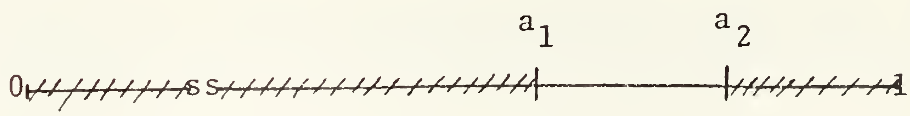
(i) Notch-gain is at least - 60 dB.

(ii) Notch-width is at least one hertz.

any coefficient, a, in)0,1(satisfies conditions (i) and (ii), and that value which produces the smoothest passband is the optimum value.

In the method of searching, first, one selects the subinterval of)0,1(such that any coefficient belonging to this subinterval will satisfy the conditions (i) and (ii); second, searching is accomplished for the minimum error function of single variable for one of the subintervals.

The subinterval can be found by computer experiment. Suppose for sixth order notch-filter one can consider Table (4-3); call (a_1, a_2) this subinterval:



As stated above, any coefficient, a, belonging to interval (a_1, a_2) will satisfy the conditions (i) and (ii).

For sixth order, the values of a_1 and a_2 are given as [see Table (4-3)]:

$$a_1 = 0.98 \quad \text{and} \quad a_2 = 0.995 .$$

For third order [see Table (4-2]:

$$a_1 = 0.98 \quad \text{and} \quad a_2 = 0.99 .$$

It is noted that the ideal (errorless) magnitude frequency response is equal to zero dB over the passband of the notch-filter. Due to the limited wordlength of computers, the actual magnitude frequency response will deviate from zero dB over the passband. If one defines:

$$\varepsilon\ (a,\ \omega_i) = 1 - H\ (a,\ \omega_i) \tag{5-1}$$

as the error function corresponding to coefficient, a, and radian frequency ω_i of sinusoidal input signal. The error function $\varepsilon\ (a,\ \omega_i)$ is defined in discrete domain which is considered as approximates the mean-squared error in continuous domain of interval between 0 and ω_i.

$$\varepsilon\ (a,\ \omega_i) \simeq \frac{1}{\omega_i} \int_0^{\omega_i} |H_\infty\ (a,\ \omega_i) - H\ (a,\ \omega_i)|^2\ d\omega_i \tag{5-2}$$

where $H_\infty\ (a,\ \omega_i)$ is ideal response and has magnitude equal to one.

Rewrite (5-2) as

$$\varepsilon\ (a,\ \omega_i) = \frac{1}{\omega_i} \int_0^{\omega_i} |1 - H\ (a,\ \omega_i)|^2\ d\omega_i \tag{5-3}$$

the system is stable, $\varepsilon\ (a,\ \omega_i)$ has to be convergent.

Let ω_i scan over the frequency range of the passband of notch-filter, the error performance or the total error function, is defined as

$$E(a) = \sum_{i=1}^{N} \varepsilon^2 (a, \omega_i)$$

or

$$E(a) = \sum_{i=1}^{N} [1 - H (a, \omega_i)]^2 \tag{5-4}$$

where $a_1 \leq a \leq a_2$ and N is a finite integer number.

The optimization problem turns out to be the least square method of single parameter. The optimum coefficient, a, is one that makes the error performance E(a) minimum. The following algorithms are described to search for the minimum of E(a). First, the interval (a_1, a_2) is subdivided into N subintervals such that

$$a_K \varepsilon \left[a_1 + \frac{(K-1)(a_2-a_1)}{N}, a_1 + \frac{K(a_2-a_1)}{N}\right]$$

where K = 1, 2, ..., N.

Then compare the values of $E(a_K)$ and $E(a_{K+1})$ corresponding to successive value of a_K and a_{K+1}, where

$$a_{K+1} \varepsilon \left[a_1 + \frac{K(a_2-a_1)}{N}, a_1 + \frac{(K+1)(a_2-a_1)}{N}\right]$$

If $E(a_K) \geq E(a_{K+1})$

the next step will compare $E(a_{K+1})$ and $E(a_{K+2})$ and store coefficient a_{K+1}.

Or if $E(a_K) \le E(a_{K+1})$, the next step will compare $E(a_K)$ and $E(a_{K+2})$ and store coefficient a_K; the execution continues until the last subinterval of (a_1, a_2) is reached.

The final result will be printed out as the optimum coefficient corresponding to the smallest value of error performance $E(a)$ for $a_1 \le a \le a_2$.

The greater the value of N, the higher accuracy one can get but the price is paid by spending more computation time.

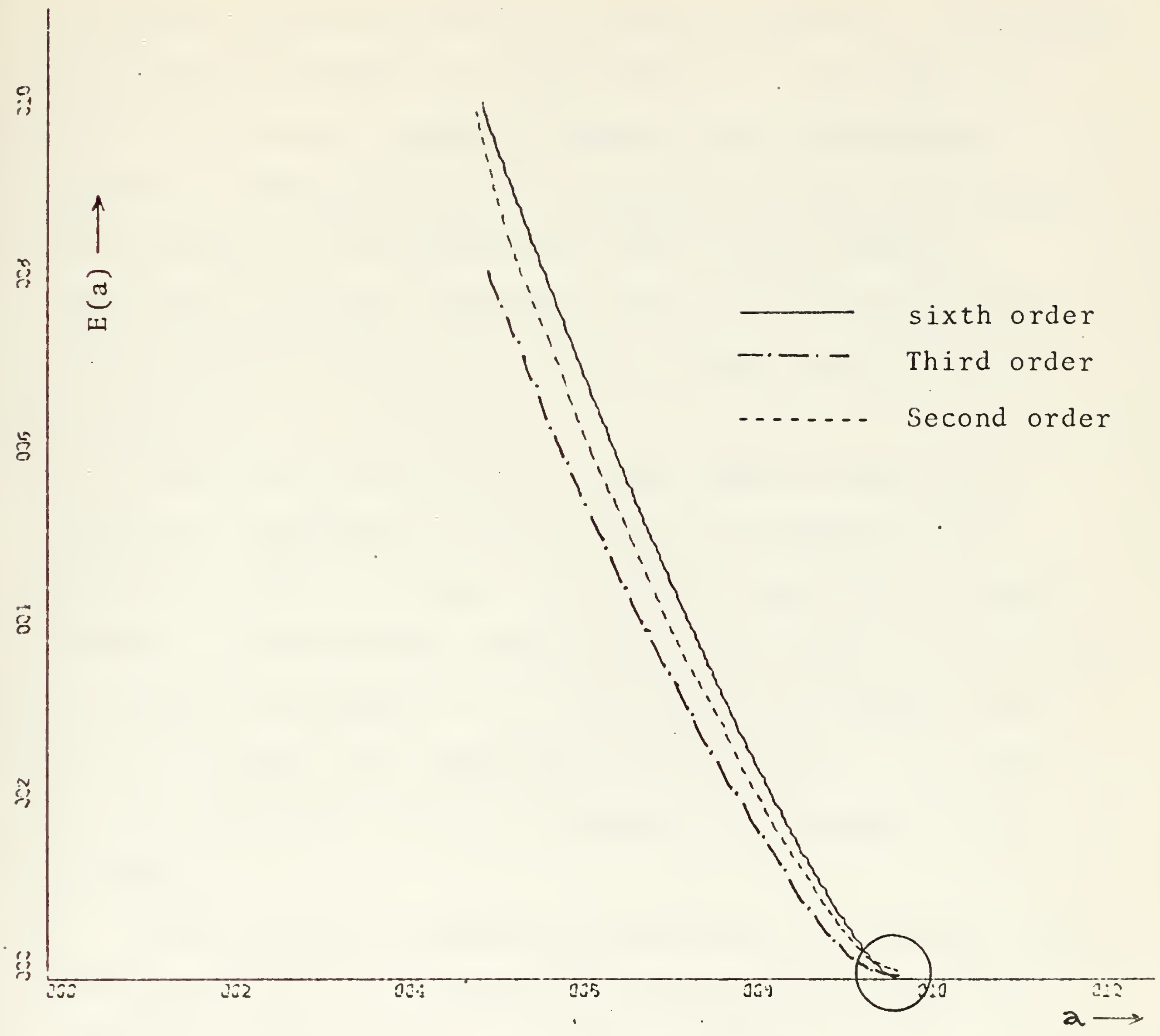
The plots of error performance are illustrated in Fig. (5-1) of second, third, and sixth order digital notch-filters. One sees that the error performance is monotonically decreasing with increasing coefficient, i.e., the greater the coefficient, the smoother the response in passband. This result can be extended for many order of digital notch-filter characterized by the transfer function (2-16). The optimum coefficient now is so easy to pick and it is the coefficient a_2.

for sixth order $a_2 = 0.995$

for third order $a_2 = 0.99$.

B. DISCUSSION AND CONCLUSION

One sees that the error performance of single parameter is unimodal. The optimum problem of this kind of function has been accomplished by many methods well-known such as Fibonacci Search, Golden Section, etc. The Fibonacci Search is known as the most efficient in optimization technique of



X-SCALE=2.00E-01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

Figure (5-1) Plot of total deviation of second, third and sixth order notch-filter between a=0.5 and a=0.999.
The region in circle is exaggerated in Fig. (5-2).

unimodal objective function of single parameter. The method used here--referred to by some authors as the uniform or factorial search--compares formatly with the Fibonacci Search in that it takes more computation time than, but in this case of error-performance E(a), in the author's opinion, is the most efficient one. In spite of the cost of computation time, the method is simple and the result is precise.

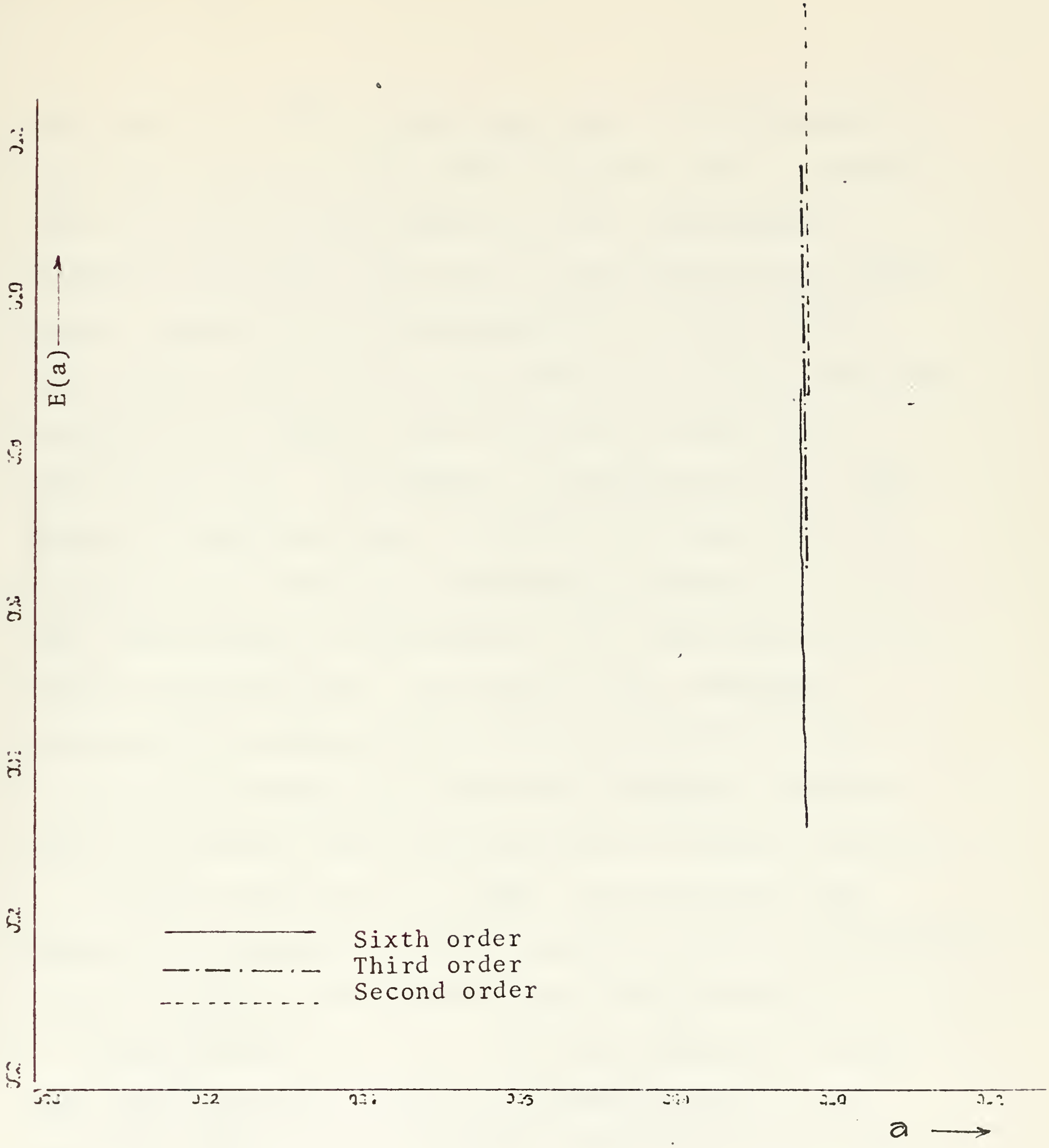
The results carried out in this chapter agree with computer experiments in Chapter IV by considering the numerical value in Tables (4-1), (4-2), and (4-3). Higher orders of notch-filter caused deeper notch-gains and smoother response in passband. It is noted that the smoother pass-band is correct for coefficient close to one. It is seen in Fig. (5-2) for noise-performance in the region close to one.

The value of the optimum coefficient, a, increases with the order of filter provided the conditions (i) and (ii) are fixed. The higher order of notch-filter, the closer a approaches one, the constraint is placed on a as mentioned in section D of Chapter II, due to the finite wordlength, e.g., the single precision on IBM 360/70 has 32-bit wordlength. The inequality (2-23) gives

$$1 - a_2 = \Delta > 2^{-31}$$

or

$$a_2 < 1 - 2^{-31} \simeq 0.999999999 \qquad (5.5)$$



X-SCALE=2.00E-01 UNITS INCH.
Y-SCALE=2.00E-03 UNITS INCH.

Figure (5-2) Noise performance in the region of coefficient close to one.

The value $(1 - 2^{-31})$ is the upper limit of coefficient, a, in theory. In practical view, the upper limit is smaller than $(1 - 2^{-31})$ by the reason of error introduced in arithmetic operation and error in representation as stated above. Besides, in the implementation of notch-filter, the optimum coefficient corresponding the larger wordlength in register of computer will be closer to unity than the shorter one. For example, in implementation of sixth order of single precision on IBM 360, the upper limit of coefficient is 0.999, for coefficient exceeds this limit, the system begins to degenerate. It is the same limit for second and third order. One can see this phenomenon numerically in Tables (4-1), (4-2), and (4-3) and the behavior of changing in steady-state frequency response plots of second, third and sixth order in previous chapter.

One sees that for a = 0.9999, the notch-gain less than 3 dB in magnitude and the passband ripple is unacceptable. It is foolish to accept them as the notch-filters.

Furthermore, for $a > (1 - 2^{31})$ the system may become the oscillator due to error in rounding since it could make the poles of their transfer functions exactly on the unit circle of z-plane.

VI. CONCLUSION

The filter design was used to design the digital notch-filters. The effect of order of filter on notch-gain, notch-width, and passband ripple was investigated.

From results of this study, the simplest configuration of digital notch-filters was found; hence implementation will be economical in hardware. The notch-frequency, the notch-gain and the notch-width are very easily controllable by varying only one coefficient in realization as well as the agreement between the predicted and practice sampling time. This property is usable in an array of notch-filters to eliminate interference signals of variety of frequencies.

It was seen that with selected coefficient the pass-band is very smooth (for sixth order less than 0.05 dB) and the notch-width is very narrow (less than one hertz). The improvement can be achieved by increasing the order of filters provided the coefficient close enough to the optimum coefficient which was found by Search procedure. The optimum computer program was used and the result was precise. This program required large computer time. That is the reason some authors do not like it.

LIST OF REFERENCES

1. B. Gold and C. M. Rader, "Digital Processing of Signal," McGraw Hill, 1969.

2. J. F. Kaiser, "Digital Filter" in System Analysis by Digital Computers, edited by F. F. Kuo, John Wiley and Sons, 1966.

3. S. R. Parker and S. Hess, "Canonic Realization of Second Order Digital Notch-Filters due to Finite Precision Arithmetic," IEEE Trans. on Circuit Theory, Vol. CT-19, No. 4, July 1973, pp. 410-413.

4. Leland Brooks Jackson, "An Analysis of Round-Off Noise in Digital Filters," Ph.D. dissertation, 1969, Stevens Institute of Technology.

5. Paul Edward Girard, "Correlated Noise Effects of Structure in Digital Filters," Ph.D. dissertation, 1974, Naval Postgraduate School.

6. James A. Cadzow, "Digital Notch-Filter Design Procedure," IEEE Trans. on Acoustics, Speech and Signal Processing, Vol. ASSP-22, No. 1, Feb. 1974, pp. 10-15.

7. Rotaro Hirano , Shotard Nishimura and Sanjit K. Mitra, "Design of Digital Notch-Filters," IEEE Trans. on Circuit and Systems, Vol. CAS-21, No. 4, July 1974, pp. 540-546.

8. R. Carney, "Design of a Digital Notch-Filter with Tracking Requirement," IEEE Trans. on Space Electronics and Telemetry, Vol. SET-9, No. 4, Dec. 1963, pp. 109-114.

9. Gabor C. Temes and Donald A. Calahan, "Computer-Aided Network Optimization--The State-of-the-Art," Proc. of IEEE, Vol. 55, No. 11, Nov. 1967, pp. 1832-1863.

10. S. L. S. Jacoby, J. S. Kowalik and J. T. Pizzo, Iterative Methods for Non-linear Optimization Problems, Prentice Hall, 1972.

11. J. B. Knowles and E. M. Olcayto, "Coefficient Accuracy and Digital Filter Response," IEEE Trans. on Circuit Theory, Vol. CT-15, No. 1, March 1968, pp. 31-41.

12. Bede Liu, "Effect of Finite Wordlength on the Accuracy of Digital Filter--A Review," IEEE Trans. on Circuit Theory, Vol. CT-18, No. 6, Nov. 1971, pp. 670-677.

13. Ernst Avenhaus, "On the Design of Digital Filters with Coefficients of Limited Wordlength," IEEE Trans. on Audio and Electroacoustics, Vol. AU-20, No. 3, August 1972, pp. 206-212.

14. Howard D. Helms, "Digital Filters with Equiripple or Minimax Responses," IEEE Trans. on Audio and Electroacoustics, Vol. AU-19, No. 1, March 1971, pp. 87-93.

15. R. E. Massara and J. K. Fidler, "Efficient Damping Method for Least-Squares Algorithms," Electronics Letters, Vol. 11, No. 2, Jan. 1975, pp. 33-34.

16. Min Soo Suk and Sanjit K. Mitra, "Computer-Aided Design of Digital Filters with Finite Wordlengths," IEEE Trans. on Audio and Electroacoustics, Vol. AU-20, No. 5, Dec. 1972, pp. 356-363.

17. Kenneth Steiglitz, "Computer-Aided Design of Recursive Digital Filter," IEEE Trans. on Audio and Electroacoustics, Vol. AU-18, No. 2, June 1970, pp. 123-129.

18. E. Avenhaus and W. Schüssler, "On the Approximation Problem in the Design of Digital Filters with Limited Wordlength," in Digital Signal Processing, edited by L. R. Rabiner and C. M. Rader, 1972, IEEE Press, pp. 383-384.

19. W. S. Dorn and D. D. McCracken, Numerical Methods with Fortran IV Case Study, John Wiley and Sons, 1972.

20. R. Edward and A. Bradley, "Design of Digital Filters by Computer," International Journal for Numerical Methods in Engineering, Vol. 2, June 1969, pp. 311-333.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Department Chairman, Code 52 Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	2
4.	Dr. Sydney R. Parker (thesis advisor) Dean, College of Engineering Busch Campus Rutgers University New Brunswick, New Jersey 98303	5
5.	Asst Professor T. F. Tao, Code 52 TV Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	1
6.	LCDR Lennart Souchon 2392 Gluecksburg Am Thingplatz 7 Germany	1
7.	LTjg Tran Truc Viet c/o Mrs. Betty Field P. O. Box BF Pacific Grove, California 93950	4